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Math 850 – Federico Ardila
Homework 1 Feb. 9, 2012

2. (Some examples of $\mathbb{F}$-algebras.) *Collaborated with Seth, Katrina and Karla.*
(a) Prove that $\mathbb{F}[x]\otimes\mathbb{F}[x] \cong \mathbb{F}[x,y]$ as $\mathbb{F}$-algebras.

Proof: Let $f: \mathbb{F}[x] \times \mathbb{F}[x] \to \mathbb{F}[x,y]$ defined by $(p(x), q(x)) \longmapsto p(x)q(y)$.

Since $\mathbb{F}[x]$ and $\mathbb{F}[x,y]$ are vector spaces $f: \mathbb{F}[x] \times \mathbb{F}[x] \to \mathbb{F}[x,y]$ is a bilinear map. Thus by the Universal Property, f descends to a linear map $\tilde{f}$: $\mathbb{F}[x]\otimes\mathbb{F}[x] \longmapsto \mathbb{F}[x,y]$.

Since $\tilde{f}$ is a linear map, for $p_i(x)\otimes q_j(x) \,\epsilon\, \mathbb{F}[x]\otimes\mathbb{F}[x]$,

$$\tilde{f}((p_1(x)\otimes q_1(x)) + (p_2(x)\otimes q_2(x))) = \tilde{f}(p_1(x)\otimes q_1(x)) + \tilde{f}(p_2(x)\otimes q_2(x))$$
$$and$$
$$\tilde{f}\big(\lambda(p_1(x)\otimes q_1(x)\big) = \lambda\tilde{f}(p_1(x)\otimes q_1(x))$$

To show $\tilde{f}$ is an algebra homomorphism, we need to show,

1) $\tilde{f}((p_1(x)\otimes q_1(x))(p_2(x)\otimes q_2(x))) = \tilde{f}(p_1(x)\otimes q_1(x))\tilde{f}(p_2(x)\otimes q_2(x))$
2) $\tilde{f}\big(1_{\mathbb{F}[x]\otimes\mathbb{F}[x]}\big) = 1_{\mathbb{F}[x,y]}$

1) $\tilde{f}((p_1(x)\otimes q_1(x))(p_2(x)\otimes q_2(x)))$
$=\tilde{f}(p_1(x))(p_2(x)\otimes q_1(x)q_2(x))$
$= (p_1(x))(p_2(x))(q_1(y)q_2(y))$
$= p_1(x)q_1(y)p_2(x)q_2(y))$
$=\tilde{f}(p_1(x)\otimes q_1(x))\tilde{f}(p_2(x)\otimes q_2(x))$

2) $\tilde{f}\big(1_{\mathbb{F}[x]\otimes\mathbb{F}[x]}\big) = \tilde{f}(1\otimes 1) = \tilde{f}(1) = 1_{\mathbb{F}[x,y]}$.

So, $\tilde{f}$ is an algebra homomorphism.
To check $\tilde{f}$ is surjective, let $\{x^n y^m : n, m \,\epsilon\, \mathbb{N}\}$ be a basis in $\mathbb{F}[x,y]$. Then $\tilde{f}(x^n\otimes x^m) = x^n y^m$. Thus $\tilde{f}$ is surjective.
Now suppose $\tilde{f}(\sum_k p_k(x)\otimes q_k(x)) = 0$.
This implies that $\sum_k p_k(x)q_k(y) = 0$, so for each k, $p_k(x) = 0$ or $q_k(y) = 0$.
If $p_k(x) = 0$, then $p_k(x)\otimes q_k(x) = \sum_k 0 \otimes q_k(x) = 0$.
And If $q_k(y) = 0$, then $\sum_k p_k(x) \otimes q_k(x) = \sum_k p_k(x)0 = 0$.
Thus $\tilde{f}$ is injective. Therefore it is an isomorphism. ∎

(b) If A is a $\mathbb{F}$-algebra, let $M_{n\times n}(A)$ be the $\mathbb{F}$-algebra of $n \times n$ matrices with entries in A. Explain briefly why $M_{n\times n}(A)$ is a $\mathbb{F}$-algebra.

Solution: Let $B = M_{n\times n}(A)$ with A a $\mathbb{F}$-algebra. Since A is a $\mathbb{F}$-algebra, we know that A is a vector space with entries in $\mathbb{F}$. So an element of A can look something like $[a_1 \;\; a_2 \;\; a_3 \;\; \dots\dots..]$

Similar to our examples during lecture 3 we can show B is a $\mathbb{F}$-algebra since there exists a map $u: A \to B$ such that $\lambda \,\epsilon\, A \longmapsto \lambda I = \begin{bmatrix} \lambda & \cdots & 0 \\ \vdots & \lambda & \vdots \\ 0 & \cdots & \lambda \end{bmatrix}$ and therefore B is a $\mathbb{F}$-algebra. ∎

Prove that $M_{n\times n}(A) \cong M_{n\times n}(\mathbb{F})\otimes A$ as $\mathbb{F}$-algebras.
Proof:
Let $f: M_{n\times n}(\mathbb{F}) \times A \longmapsto M_{n\times n}(A)$ defined by

$$\left(\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}, a\right) \longmapsto \begin{bmatrix} ax_{11} & \cdots & ax_{1n} \\ \vdots & \ddots & \vdots \\ ax_{n1} & \cdots & ax_{nn} \end{bmatrix}$$

We see that this map is bilinear since,

$$f\left(\left[x_{ij}\right]_{n\times n} + \left[y_{ij}\right]_{n\times n}, a\right) = f\left(\left[x_{ij} + y_{ij}\right]_{n\times n}, a\right) = \left[x_{ij}a + y_{ij}a\right]_{n\times n} = \left[x_{ij}a\right]_{n\times n} + \left[y_{ij}a\right]_{n\times n}$$
$$= f\left(\left[x_{ij}\right]_{n\times n}, a\right) + f\left(\left[y_{ij}\right]_{n\times n}, a\right)$$

and

$$f\left(\left[x_{ij}\right]_{n\times n}, a+b\right) = \left[x_{ij}(a+b)\right]_{n\times n} = \left[x_{ij}a + x_{ij}b\right]_{n\times n} = \left[x_{ij}a)\right]_{n\times n} + \left[x_{ij}b\right]_{n\times n}$$
$$= f\left(\left[x_{ij}\right]_{n\times n}, a\right) + f\left(\left[x_{ij}\right]_{n\times n}, b\right)$$

So by the Universal Property, f descends to $\tilde{f}$.

$$\tilde{f}\colon M_{n\times n}(\mathbb{F}) \otimes A \cong M_{n\times n}(A)$$

Since $\tilde{f}$ is a linear map,

$$\tilde{f}\left(\left(\left[x_{ij}\right]_{n\times n}\otimes a\right) + \left(\left[y_{ij}\right]_{n\times n}\otimes b\right)\right) = \tilde{f}\left(\left[x_{ij}\right]_{n\times n}\otimes a\right) + \tilde{f}\left(\left[y_{ij}\right]_{n\times n}\otimes b\right)$$

and

$$\tilde{f}\left(\lambda\left[x_{ij}\right]_{n\times n}\otimes a\right) = \lambda\tilde{f}\left(\left[x_{ij}\right]_{n\times n}\otimes a\right).$$

Now

$$\tilde{f}\left(\left(\left[x_{ij}\right]_{n\times n}\otimes a\right)\left(\left[y_{ij}\right]_{n\times n}\otimes b\right)\right) = \tilde{f}\left(\left[x_{ij}y_{ij}\right]_{n\times n}\otimes ab\right) = \left[x_{ij}y_{ij}ab\right]_{n\times n} = \left[x_{ij}a\right]_{n\times n}\left[y_{ij}b\right]_{n\times n}$$
$$= \tilde{f}\left(\left[x_{ij}\right]_{n\times n}\otimes a\right)\tilde{f}\left(\left[y_{ij}\right]_{n\times n}\otimes b\right)$$

and

$$\tilde{f}\left(1_{M_{n\times n}(\mathbb{F})\otimes A}\right) = \tilde{f}([1]_{n\times n}\otimes 1_A) = [1\cdot 1_A]_{n\times n} = 1_{M_{n\times n}(A).}$$

Thus $\tilde{f}$ is a homomorphism.

$\ker\tilde{f} = \left\{\left[x_{ij}\right]_{n\times n}\otimes a \in M_{n\times n}(\mathbb{F})\otimes A : \tilde{f}\left(\left[x_{ij}\right]_{n\times n}\otimes a\right) = 1_{M_{n\times n}(A).}\right\}$

But $\tilde{f}\left(\left[x_{ij}\right]_{n\times n}\otimes a\right) = f\left(\left[x_{ij}\right]_{n\times n}, a\right) = \left[x_{ij}a\right]_{n\times n}$

So $\tilde{f}\left(\left[x_{ij}\right]_{n\times n}\otimes a\right) = 1 \Rightarrow \left[x_{ij}a\right]_{n\times n} = 1$

$$\Rightarrow \left[x_{ij}\right]_{n\times n} = 1 \text{ and } a = 1_A$$
$$\Rightarrow \left[x_{ij}\right]_{n\times n}\otimes a = [1]_{n\times n}\otimes 1_A = 1_{M_{n\times n}(\mathbb{F})\otimes A}$$

Thus $\tilde{f}$ is injective.

To show surjectivity, let $\{[e_i]_{n\times n} : i \in I\}$ be a basis for $M_{n\times n}(\mathbb{F})$. Then $\{[e_i a]_{n\times n} : i \in I\}$ is a basis for $M_{n\times n}(A)$. But $[e_i a]_{n\times n} = f([e_i]_{n\times n}, a) = \tilde{f}([e_i]_{n\times n}\otimes a)$. Therefore, $\tilde{f}$ is an isomorphism. ■

(c) Prove that $M_{m\times m}(\mathbb{F})\otimes M_{n\times n}(\mathbb{F}) \cong M_{mn\times mn}(\mathbb{F})$ as $\mathbb{F}$-algebras.

<u>Proof:</u> Let $f\colon M_{m\times m}(\mathbb{F})\times M_{n\times n}(\mathbb{F}) \longmapsto M_{mn\times mn}(\mathbb{F})$ defined by

$$\left(\begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mm}\end{bmatrix}, \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn}\end{bmatrix}\right) \longmapsto \begin{bmatrix} x_{11}\begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn}\end{bmatrix} & \cdots & x_{1m}\begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn}\end{bmatrix} \\ \vdots & \ddots & \vdots \\ x_{m1}\begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn}\end{bmatrix} & \cdots & x_{mm}\begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn}\end{bmatrix}\end{bmatrix}$$

f is a binlinear map since adding 2 matrices is adding each of the corresponding entries. So by the Universal Property, f can descend to $\tilde{f}\colon M_{n\times n}(\mathbb{F})\otimes A \cong M_{n\times n}(A)$.

Since $\tilde{f}$ is linear,

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$$\tilde{f}\left(\left([x_{ij}]_{m\times m}\otimes[y_{kl}]_{n\times n}\right)+\left([u_{ij}]_{m\times m}\otimes[v_{kl}]_{n\times n}\right)\right)$$
$$=\tilde{f}\left([x_{ij}]_{m\times m}\otimes[y_{kl}]_{n\times n}\right)+\tilde{f}\left([u_{ij}]_{m\times m}\otimes[v_{kl}]_{n\times n}\right)$$

and

$$\tilde{f}\left(\lambda\left([x_{ij}]_{m\times m}\otimes[y_{kl}]_{n\times n}\right)\right)=\lambda\tilde{f}\left([x_{ij}]_{m\times m}\otimes[y_{kl}]_{n\times n}\right).$$

To check surjectivity we check:

1) $$\tilde{f}\left(\left([x_{ij}]_{m\times m}\otimes[y_{kl}]_{n\times n}\right)\left([u_{ij}]_{m\times m}\otimes[v_{kl}]_{n\times n}\right)\right)=\tilde{f}\left([x_{ij}u_{ij}]_{m\times m}\otimes[y_{kl}v_{kl}]_{n\times n}\right)$$
$$=\begin{bmatrix} x_{11}u_{11}(y_{kl}v_{kl}) & \cdots & x_{1m}u_{1m}(y_{kl}v_{kl}) \\ \vdots & \ddots & \vdots \\ x_{m1}u_{m1}(y_{kl}v_{kl}) & \cdots & x_{mm}u_{mm}(y_{kl}v_{kl}) \end{bmatrix}_{mn\times mn}$$
$$=\begin{bmatrix} x_{11}(y_{kl}) & \cdots & x_{1m}(y_{kl}) \\ \vdots & \ddots & \vdots \\ x_{m1}(y_{kl}) & \cdots & x_{mm}(y_{kl}) \end{bmatrix}\begin{bmatrix} u_{11}(v_{kl}) & \cdots & u_{1m}(v_{kl}) \\ \vdots & \ddots & \vdots \\ u_{m1}(v_{kl}) & \cdots & u_{mm}(v_{kl}) \end{bmatrix}$$
$$=\tilde{f}\left([x_{ij}]_{m\times m}\otimes[y_{kl}]_{n\times n}\right)\cdot\tilde{f}\left([u_{ij}]_{m\times m}\otimes[v_{kl}]_{n\times n}\right)$$

2) $$\tilde{f}\left(1_{M_{m\times m}(\mathbb{F})\otimes M_{n\times n}(\mathbb{F})}\right)=\tilde{f}([1]_{m\times m}\otimes[1]_{n\times n})=f([1]_{mn\times mn})=1_{M_{mn\times mn}(\mathbb{F})}$$

Finally we show $\tilde{f}$ is an isomorphism. ■