1. (With Nino, Brian.) Let V and W be $\mathbb{F}$-vector spaces. We prove that if $\{v_i : i \in I\}$ and $\{w_j : j \in J\}$ are bases for V and W, respectively, then $\mathcal{B} = \{v_i \otimes w_j : i \in I; j \in J\}$ is a basis for $V \otimes W$.

Proof. First we will prove that $\mathcal{B}$ spans $V \otimes W$. We observe that it suffices to consider pure tensors, that is, the fact that the result holds for pure tensors implies that it also holds non-pure tensor elements.

Let $v \otimes w \in V \otimes W$. Let $v = \sum_{l \in L} a_l v_l$, for $0 < |L| < \infty$, and $w = \sum_{m \in M} b_m w_m$, for $0 < |M| < \infty$. Therefore,

$$v \otimes w = \left(\sum_{l \in L} a_l v_l \right) \otimes \left(\sum_{m \in M} b_m w_m \right) = \sum_{l,m} (a_l v_l \otimes b_m w_m) = \sum_{l,m} a_l b_m \left(v_l \otimes w_m \right),$$

which implies that $\mathcal{B}$ spans $V \otimes W$.

Now we will show that $\mathcal{B}$ is linearly independent. By way of contradiction, assume that $\mathcal{B}$ is linearly dependent. Let $N \subseteq \mathbb{N}$, with $0 < |N| < \infty$, and $\{a_1, a_2, ..., a_N\}$ be such that

$$\sum_{n \in N} a_n (v_n \otimes w_n) = 0,$$

where $a_i \neq 0$ for at least one $i \in N$. The above implies that $\sum_{n \in N} (a_n v_n \otimes w_n) = 0$.

By the "useful lemma," $a_n v_n = 0$ for all $n \in N$. Since $\{v_i : i \in I\}$ is a basis for V, $\{v_i : i \in I\}$ is linearly independent, so $a_n v_n = 0$ if and only if $a_n = 0$ for all $n \in N$. However, this contradicts our assumption that there is at least one $i \in N$ with $a_i \neq 0$.

We conclude that $\mathcal{B}$ is a basis for $V \otimes W$.

We observe that if $\dim V$ and $\dim W$ are finite, then $\dim(V \otimes W) = \dim V \dim W$ because there are $\dim V$ possibilities for the first coordinate, while there are $\dim W$ different possibilities for the second coordinate, which implies there are $\dim V \dim W$ different pure tensors $v_i \otimes w_j$ in the basis $\{v_i \otimes w_j : i \in I; j \in J\}$.

□