hopf algebras and combinatorics

federico ardila

san francisco state university
universidad de los andes

homework four . due tuesday mar 27 at 11:59pm (sf time = bog time – 3:00)

Note. You are encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)
You must turn in your homework – **in one .pdf file** by email at `hopfcombinatorics@gmail.com`.

In the problems below, $\mathbb{K}$ denotes an arbitrary field.

1. (Exercises on posets.)
   (a) Prove that the product of two Boolean posets is a Boolean poset.
   (b) Let a *box poset* be a finite product of finite chains. Prove that any interval of a box poset is a box poset.
2. (Computing antipodes.)
   (a) Compute the antipode of the *Hopf algebra of symmetric functions* described in Lectures 14 and 15.
   (b) Compute the antipode of the *Hopf algebra of non-commutative symmetric functions* described in Lecture 15.

   As their names suggest, and we will see soon, these two Hopf algebras are more commonly described in a different but equivalent way. In this problem, I am asking you to compute their antipodes starting from their description as incidence Hopf algebras.
3. (Antipodes of incidence Hopf algebras have order 2.)
   Prove that the antipode S of any incidence Hopf algebra satisfies $S \circ S = I$.
4. (Tensor, symmetric, and exterior Hopf algebras?)
   (a) Define a product on the tensor algebra $T(V)$ of V by setting

   $$(x_1 \otimes \cdots \otimes x_m)(y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n$$

   for any pure tensors $x_1 \otimes \cdots \otimes x_m$ and $y_1 \otimes \cdots \otimes y_n$ in $T(V)$ $(x_1, \ldots, x_m, y_1, \ldots, y_n \in V)$. Define a coproduct on $T(V)$ by setting

   $$\Delta(x) = \sum_{S \subseteq [n]} x|_S \otimes x|_{[n]\setminus S} \in T(V) \otimes T(V)$$

   for $x = x_1 \otimes \cdots \otimes x_n \in T(V)$. Here $x|_A = x_{a_1} \otimes \cdots \otimes x_{a_k}$ if $A = \{a_1 < \ldots < a_k\} \subseteq [n]$. Define a unit by $u(1) = 1$ and a counit by letting $\epsilon(x_1 \otimes \cdots \otimes x_m)$ be 1 if $m = 0$ and 0 otherwise.

   Does this turn $T(V)$ into a Hopf algebra? If so, compute its antipode.

   (b) Does the Hopf algebra structure of $T(V)$ descend to a quotient Hopf algebra structure on the exterior algebra $\bigwedge(V)$? On the symmetric algebra $S(V)$?

5. (Nice posets) Let P be a graded poset with rank function $r : P \to \mathbb{N}$. Let a_i be the number of order ideals of P of size i. The *rank-generating function* of $J(P)$ is $F(J(P), q) := \sum_{i \geq 0} a_i q^i$.

   We say P is *nice* if

$$\sum_{i \geq 0} a_i q^i = \prod_{p \in P} \frac{1 - q^{r(p)+2}}{1 - q^{r(p)+1}}$$

   Let $\mathbf{n}$ represent the n-element chain for $n \in \mathbb{Z}_{>0}$. Prove that the following posets are nice:

   (a) $P = J(\mathbf{2} \times \mathbf{n})$ for $n \in \mathbb{Z}_{>0}$.

   (b) $P = \mathbf{m} \times \mathbf{n}$ for $m, n \in \mathbb{Z}_{>0}$.

6. (Bonus problem: Nice and not so nice posets.)

   (a) Prove that $P = \mathbf{l} \times \mathbf{m} \times \mathbf{n}$ is nice for $l, m, n \in \mathbb{Z}_{>0}$. (Bonus problem: This has been solved, but there is no known "easy" proof.)

   (b) Prove that $P = J(\mathbf{3} \times \mathbf{n})$ is nice for $n \in \mathbb{Z}_{>0}$. (Bonus problem: This was a conjecture for many years. It was recently solved, but the only known proof is very difficult and computationally intensive.)

   (c) The posets $\mathbf{k} \times \mathbf{l} \times \mathbf{m} \times \mathbf{n}$ and $J(\mathbf{4} \times \mathbf{n})$ are not necessarily nice. Is there a reasonable formula for $\sum a_i q^i$ in those cases? (Bonus problem: This is unsolved.)