hopf algebras and combinatorics

federico ardila

san francisco state university
universidad de los andes

homework three . due tuesday mar 6 at 11:59pm (sf time = bog time – 3:00)

Note. You are encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)
You must turn in your homework – **in one .pdf file** by email at `hopfcombinatorics@gmail.com`.

In the problems below, $\mathbb{K}$ denotes an arbitrary field.

1. (Möbius inversion.)
   Let P be a finite poset and $A(P)$ be its incidence algebra. Recall that the identity is the function $\mathbf{1} \in A(P)$ defined by $\mathbf{1}([x,x]) = 1$ for all x in P and $\mathbf{1}([x,y]) = 0$ for all $x < y$ in P. Define the *zeta function* of P by $\zeta([x,y]) = 1$ for all $x \leq y$ in P.

   (a) Prove that $f \in A(P)$ has a two-sided inverse if and only if $f([x,x]) \neq 0$ for all $x \in P$.

   (b) By part (a), $(2-\zeta)$ is invertible in $A(P)$. Prove that
   $$(2-\zeta)^{-1}([x,y]) = (\# \text{ of chains in } P \text{ from } x \text{ to } y).$$

   (c) By part (a), ζ is invertible in $A(P)$. Its inverse $\mu = \zeta^{-1}$ is called the *Möbius function* of P. Prove the *Möbius inversion formula*:

   Let $f : P \to V$ and $g : P \to V$ be functions from P to a vector space V. Then
   $$g(y) = \sum_{x \leq y} f(x) \qquad \text{for all } y \in P$$
   if and only if
   $$f(y) = \sum_{x \leq y} \mu(x,y) g(x) \qquad \text{for all } y \in P.$$

2. (A non-cocommutative coalgebra generated by cocommutative elements.)
   Let $\mathbb{H}$ be the quaternions, regarded as a 4-dimensional algebra over $\mathbb{R}$. Let $A = \mathbb{H}^*$ be the dual coalgebra. Find a cocommutative element $a \in A$ such that the subcoalgebra of A generated by a is not cocommutative.

   The *quaternions* are the 4-dimensional algebra $\mathbb{H}$ over $\mathbb{R}$ generated by elements $1, i, j,$ and k subject to the relations $i^2 = j^2 = k^2 = ijk = -1$. You should convince yourself that $\mathbb{H}$ is indeed 4-dimensional, but you don't need to prove it.

3. (The chest trap, the knee trap, the backheel trap.)
   Prove the following identities for f, g, h in a Hopf algebra.

   (a) $\displaystyle\sum_{(h)} h_{(1)} S(h_{(2)}) \otimes h_3 = h$

   (b) $\displaystyle\sum_{(g),(h)} h_{(1)} S(g_{(1)} f h_{(2)}) g_{(2)} = \epsilon(gh) S(f)$

   (c) $\displaystyle\sum_{(h)} (1 \otimes S(h_{(3)}) h_{(1)}) \Delta S(h_{(2)}) = (S \otimes S)\Delta(h)$

4. (A Hopf algebras with antipode of degree $2n$.)
Let $F = \mathbb{K}\{x, y, z\}$ be the free non-commutative algebra on three variables. Define a coproduct and counit on F by defining

$$\Delta(x) = x \otimes x, \qquad \Delta(y) = y \otimes y, \qquad \Delta(z) = 1 \otimes z + z \otimes x$$

$$\epsilon(x) = 1, \qquad \epsilon(y) = 1, \qquad \epsilon(z) = 0,$$

for the generators, and extending these maps linearly and multiplicatively.

   (a) Verify that Δ and ϵ turn F into a bialgebra.

   (b) Prove that the two-sided ideal $\langle xy-1, yx-1\rangle$ is a biideal of F, and therefore the quotient $H = F/\langle xy-1, yx-1\rangle$ is a bialgebra.

   (c) Prove that H is a Hopf algebra by finding its (unique) antipode S. Find the order of S.

   (d) Prove that the two-sided ideal $\langle x^n - 1\rangle$ is a Hopf ideal of H, and therefore $J = H/\langle x^n - 1\rangle$ is a Hopf algebra.

   (e) Prove that the antipode of J has order $2n$.

5. (Hopf algebras of permutations and graphs)

   (a) Recall that we defined the *Hopf algebra of graphs* on the linear span of the isomorphism classes of finite simple graphs.[1] The product $G \cdot H$ is the disjoint union of G and H. The coproduct of a graph G on vertex set V is $\Delta(G) = \sum_{S \subseteq V} G|_S \otimes G|_{V\setminus S}$. Here $G|_A$ denotes the induced subgraph of G with vertex set A. The unit is given by $u(1) = \emptyset$, the graph with no vertices. The counit is given by $\epsilon(\emptyset) = 1$ and $\epsilon(G) = 0$ for all $G \neq \emptyset$.

   Vertify that this is indeed a Hopf algebra. Find a simple formula for the antipode.

   (b) Now we wish to define a second *Hopf algebra of permutations* on the linear span of all permutations of $[n] = \{1, \ldots, n\}$ for all $n \geq 0$ – different from the one defined on the second day of class. The product of the permutations $a_1, \ldots, a_n$ of $[n]$ and $b_1, \ldots, b_m$ of $[m]$ (in one-line notation) is the permutation $a_1, \ldots, a_n, b_1 + n, \ldots, b_m + n$ of $[n+m]$. The coproduct of $\pi \in S_n$ is $\Delta(\pi) = \sum_{A \subseteq [n]} \mathrm{st}(\pi|_A) \otimes \mathrm{st}(\pi|_{[n]\setminus A})$. Here $\mathrm{st}(\pi|_B)$ denotes the permutation of $|B|$ whose entries are in the same relative order as the entries of B inside π.[2] The unit is given by $u(1) = \emptyset$, the permutation of the empty set. The counit sends the permutation $\emptyset$ to 1 and all other permutations to 0.

   Is this indeed a Hopf algebra? If so, find a simple formula for the antipode.

6. (Bonus problem: Addition formulas.)
We say that a formal power series $F(x, y) \in \mathbb{C}[[x, y]]$ is an *addition formula* for the formal power series $f(x) = x+$ (higher order terms) $\in \mathbb{C}[[x]]$ if and only if $f(x+y) = F(f(x), f(y))$.

   (a) Prove that any $f(x) = x+$ (higher order terms) $\in \mathbb{C}[[x]]$ has a unique addition formula.

   (b) Prove that $F(x, y)$ is an addition formula if and only if $F(x, y) = x+y+$ (higher order terms) and $F(x, F(y, z)) = F(F(x, y), z)$.

   (c) For each of the following functions $F(x, y)$, find a function $f(x)$ for which $F(x, y)$ is an addition formula:

$$x + y, \qquad x + y + xy, \qquad \frac{x+y}{1-xy}, \qquad x\sqrt{1-y^2} + y\sqrt{1-x^2}.$$

[1] A graph is *simple* if there are no "loops" (edges connecting a vertex to itself) or "parallel edges" (several edges connecting the same two vertices).

[2] For instance, if $\pi = 2634715$ then the restriction to $\{2, 4, 6, 7\}$ is $\pi|_{\{2,4,6,7\}} = 2647$ and its standardization is $\mathrm{st}(\pi|_{\{2,4,6,7\}}) = 1324$.