# hopf algebras and combinatorics

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## homework two . due thursday feb 23 at 11:59pm (sf time = bog time – 3:00)

Note. You are encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)
You must turn in your homework – **in one .pdf file** by email at `hopfcombinatorics@gmail.com`.

In the problems below, $\mathbb{F}$ denotes an arbitrary field.

1. (A non-commutative, non-cocommutative bialgebra, continued.)
   Consider the $\mathbb{F}$-algebra $H_4$ generated by indeterminates $g$ and $x$ subject to the relations $g^2 = 1, x^2 = 0$, and $xg = -gx$. Prove that the coproduct given by $\Delta g = g \otimes g$ and $\Delta x = x \otimes 1 + g \otimes x$ and the counit given by $\epsilon(g) = 1$ and $\epsilon(x) = 0$ turn $H_4$ into a bialgebra.
2. (Practice with Sweedler notation.) [1]
   (a) Let $T : C \otimes C \to C \otimes C$ be the *twist* map given by $T(c \otimes d) = d \otimes c$ for $c, d \in C$. Write $T\Delta(c)$ in Sweedler notation.
   (b) Prove the following identities:
      i. $\Delta(c) = \sum_{(c)} \epsilon(c_{(2)}) \otimes \Delta(c_{(1)})$
      ii. $\Delta(c) = \sum_{(c)} c_{(1)} \otimes \epsilon(c_{(3)}) \otimes c_{(2)}$
      iii. $c = \sum_{(c)} \epsilon(c_{(1)}) \otimes \epsilon(c_{(3)}) \otimes c_{(2)}$
3. (The dual algebra of a coalgebra is indeed an algebra.)
   Let $(C, \Delta, \epsilon)$ be a coalgebra. Prove that $(C^*, m, u)$, as defined in class, is an algebra.
4. (Subcoalgebras of the set coalgebra.)
   For a set $S$, the vector space $\mathbb{F}S$ has a coalgebra structure given by the coproduct $\Delta(s) = s \otimes s$ and the counit $\Delta(s) = 1$ for all $s \in S$. Find all the subcoalgebras of $\mathbb{F}S$.
5. (Ideals of indicence algebras)
   (a) For a finite poset $P$, regard the set $\text{Int}(P) = \{[x, y] : x \leq y \in P\}$ as a poset ordered by containment. Let $J^o(\text{Int}(P))$ be the poset of downsets of $\text{Int}(P)$ ordered by reverse containment.
   Prove that the poset of ideals of the incidence algebra $I(P)$ is isomorphic to $J^o(\text{Int}(P))$.
   (b) Find the number of ideals of $I(C_n)$ and $I(A_n)$, the incidence algebras of the $n$-element chain $C_n$ and the $n$-element antichain $A_n$.

   (A *downset* or *order ideal* of a poset $Q$ is a subset $I \subseteq Q$ such that $x \leq y$ and $y \in I$ imply that $x \in I$.) A *chain* is a poset where every two elements are comparable (and hence the poset is totally ordered.). An *antichain* is a poset where no two elements are comparable.
6. (Bonus problem: Incidence (co)algebras distinguish posets.)
   If two finite posets $P$ and $Q$ have isomorphic incidence coalgebras over $\mathbb{F}$, prove that they are isomorphic.

[1] To score beautiful goals, we better practice trapping the ball first.