hopf algebras and combinatorics

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homework one . due tuesday feb 7 at 11:59pm (sf time = bog time – 3:00)

Note. You are encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)
You must turn in your homework – **in one .pdf file** by email at `hopfcombinatorics@gmail.com`.

In the problems below, $\mathbb{F}$ denotes an arbitrary field and G denotes an arbitrary group.

1. (Bases of tensor products.)
   Let V and W be $\mathbb{F}$-vector spaces. If $\{v_i : i \in I\}$ is a basis for V and $\{w_j : j \in J\}$ is a basis for W, prove that $\{v_i \otimes w_j : i \in I, j \in J\}$ is a basis for $V \otimes W$. Conclude that, if $\dim V$ and $\dim W$ are finite, then $\dim V \otimes W = \dim V \dim W$.
2. (Some examples of $\mathbb{F}$-algebras.)
   (a) Prove that $\mathbb{F}[x] \otimes \mathbb{F}[x] \cong \mathbb{F}[x, y]$ as $\mathbb{F}$-algebras.
   (b) If A is a $\mathbb{F}$-algebra, let $M_{n\times n}(A)$ be the $\mathbb{F}$-algebra of $n \times n$ matrices with entries in A. Explain briefly why $M_{n\times n}(A)$ is a $\mathbb{F}$-algebra. Prove that $M_{n\times n}(A) \cong M_{n\times n}(\mathbb{F}) \otimes A$ as $\mathbb{F}$-algebras.
   (c) Prove that $M_{n\times n}(\mathbb{F}) \otimes M_{n\times n}(\mathbb{F}) \cong M_{mn\times mn}(\mathbb{F})$ as $\mathbb{F}$-algebras.
3. (Grouplike elements in the group ring.)
   In a coalgebra, we say the element x is *grouplike* if $\Delta(x) = x \otimes x$. Prove that in the group ring $\mathbb{F}[G]$, x is grouplike if and only if $x \in G$.
4. (A non-commutative, non-cocommutative bialgebra.)
   Let $q \in \mathbb{F}$ be nonzero. Consider the $\mathbb{F}$-algebra H_4 generated by indeterminates g and x subject to the relations $g^2 = 1, x^2 = 0$, and $xg = -gx$.
   (a) Show that $1, g, x$, and gx form a basis for H_4.
   (b) Express the product $(a + bg + cx + dgx)(a' + b'g + c'x + d'gx)$ in terms of this basis, where $a, b, c, d, a', b', c', d' \in \mathbb{F}$.
   (c) Show that the coproduct given by $\Delta g = g \otimes g$ and $\Delta x = x \otimes 1 + g \otimes x$ and the counit given by $\epsilon(g) = 1$ and $\epsilon(x) = 0$ turn H_4 into a bialgebra.
   (d) Express the coproduct $\Delta(a + bg + cx + dgx)$ in terms of this basis.
5. (Tensor, symmetric, and exterior algebras.)
   Let V be a given d-dimensional $\mathbb{F}$-vector space for some $d \in \mathbb{N}$.
   (a) Let $V^{\otimes k} = V \otimes \cdots \otimes V$ (where there are k copies of V) be the *kth tensor product of* V, and let
   $$T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{F} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$$
   be the *tensor algebra* of V. Find its *Hilbert series* $\mathrm{Hilb}(T(V); q) = \sum_{n=0}^{\infty} \dim(V^{\otimes k})q^k$.
   (b) Let
   $$S(V) = T(V)/\langle u \otimes v - v \otimes u \,:\, u, v \in V\rangle$$
   be the *symmetric algebra* of V. Find its Hilbert series.
   (c) Let
   $$S(V) = T(V)/\langle v \otimes v \,:\, v \in V\rangle$$
   be the *exterior algebra* of V. Find its Hilbert series.

6. (Bonus problem 1: Finite generation of algebras.)
   Find an example of two $\mathbb{F}$-algebras $A \subset B$ such that B is finitely generated and A is not finitely generated.

   (We say an algebra A over $\mathbb{F}$ is finitely generated if there exist $a_1, \ldots, a_n \in A$ such that A is generated as a ring by $a_1, \ldots, a_n$, and $\mathbb{F}$.)

7. (Bonus problem 2: A "Catalan algebra".)
   Let $q \in \mathbb{F}$ be nonzero. Consider the $\mathbb{F}$-algebra A_n with generators $e_1, \ldots, e_{n-1}$ subject to the relations

$$\begin{aligned} e_i^2 &= e_i && \text{for all } i, \\ e_i e_j &= e_j e_i && \text{for all } i, j \text{ with } |i-j| > 1, \\ e_i e_j e_i &= q e_i && \text{for all } i, j \text{ with } |i-j| = 1. \end{aligned}$$

   Prove that the dimension of A_n is the nth *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.