Lecture 9
10.01.13

Prop $$\frac{1}{\prod_{i\geq 1}(1-tx^i)} = \sum_{k\geq 0} \frac{t^k x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$$

Pf Note that

$$LHS = (1+tx^1+t^2x^{1+1}+\dots)(1+tx^2+t^2x^{2+2}+\dots)(1+tx^3+t^2x^{3+3}+\dots)\dots$$

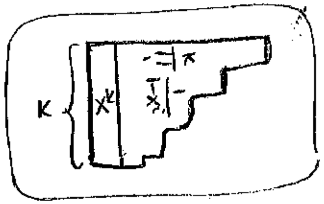
$$= \sum_{\lambda} t^{\ell(\lambda)} x^{|\lambda|}$$

Now if we look at λ such that $\ell(\lambda)=k$.

$$\sum_{\lambda:\ell(\lambda)=k} x^{|\lambda|} = \sum_{\lambda:\ell(\lambda)\leq k} x^{|\lambda|} - \sum_{\lambda:\ell(\lambda)\leq k-1} x^{|\lambda|}$$

$$= \frac{1}{(1-x)\cdots(1-x^k)} - \frac{1}{(1-x)\cdots(1-x^{k-1})}$$

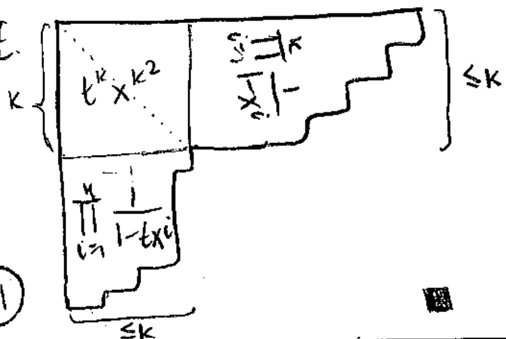
$$= \frac{x^k}{(1-x)\cdots(1-x^k)} \quad \blacksquare$$



Pictorially

Prop $$\frac{1}{\prod_{i\geq 1}(1-tx^i)} = \sum_{k\geq 0} \frac{t^k x^{k^2}}{(1-x)(1-x^2)\cdots(1-x^k)(1-tx)(1-tx^2)\cdots(1-tx^k)}$$

Pf.



$\blacksquare$

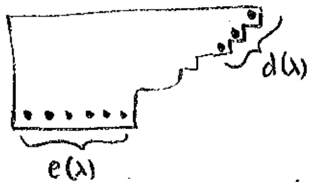
Prop (Euler's pentagonal number theorem)

$$\prod_{k\geq 1}(1-x^k) = \sum_{n\in\mathbb{Z}} (-1)^n x^{n(3n-1)/2}$$

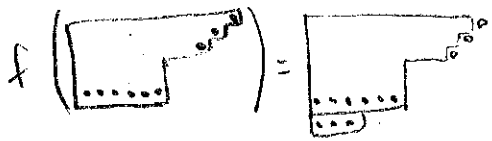
$$= 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\dots$$

Pf. The coeff of x^n is $\sum_{\substack{\lambda\vdash n \\ \text{distinct parts}}} (-1)^{\ell(\lambda)}$. This is usually 0, so we can try to pair up even+odd λ:

Let $S=\{\lambda\vdash n$ of distinct parts$\}$. For $\lambda\in S$ let



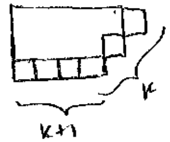
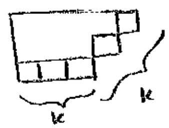
a) If $d(\lambda)<e(\lambda)$ let



b) If $d(\lambda)\geq e(\lambda)$ let

Notice that • $f(f(\lambda))=\lambda$ "sign-reversing involution".
• $\ell(f(\lambda))=\ell(\lambda)+1$

Only problem: $f(\lambda)\notin S$ if

$\lambda =$  or $\lambda =$ 

$|\lambda| = k^2+k(k+1)/2 = -k(-3k-1)/2$ $\quad$ $|\lambda| = k^2+k(k-1)/2 = k(3k-1)/2$

Remark

Since $\prod_{k\geq 1} \frac{1}{1-x^k} = \sum_{n\geq 0} p(n)x^n$

$$\prod_{k\geq 1}(1-x^k) = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\dots$$

when we multiply these and compare coeffs of x^n we get

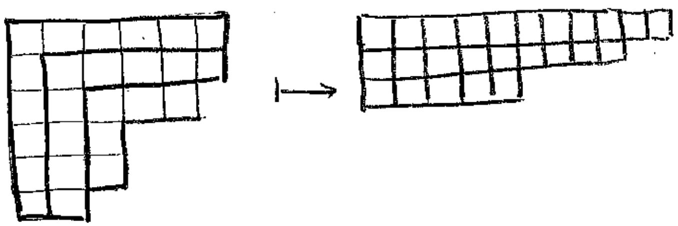
$$0 = p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-p(n-12)\cdots$$

This recurrence is the best way to date to compute $p(1), p(2), \dots$ There are other ways of computing $p(n)$ only. Also $p(n) \sim e^{\pi\sqrt{2n/3}}/4\sqrt{3}\,n$

(Compare with $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$)

Prop The # of self-conjugate partitions of n equals the # of partitions of n into odd parts

Pf



$6+6+5+3+3+2 \longmapsto 11+9+5$ ∎

Formal Power Series

Now that we've played enough with formal power series to know what we might need to worry about, let's discuss why we don't need to worry.

Let R = comm. ring. (For us usually $R = \mathbb{R}$ or $\mathbb{C}$)

A <u>formal power series</u> is a sequence

$(a_0, a_1, a_2, \dots)$ which we write "$a_0 + a_1x + a_2x^2 + \dots = A(x)$"
$(a_i \in R)$ Write $a_n = [x^n]A(x)$

The ring of formal power series $R[[x]]$ has ops

$+: (a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{Z}} = (a_n+b_n)_{n\in\mathbb{Z}}$

$\cdot: (a_n)_{n\in\mathbb{N}} \cdot (b_n)_{n\in\mathbb{Z}} = (a_0b_n + a_1b_{n-1} + \dots + a_nb_0)_{n\in\mathbb{Z}}$

(Consistent with our power series notation)

We have $0 = (0,0,\dots)$, $1 = (1,0,0,\dots)$

Easy: assoc. of $+$, of $\cdot$
comm. of $+$, of $\cdot$
distrib of $+,\cdot$

See: EC1, Sec 1.1
Ivan Niven "Formal Power Series" (Amer Math Monthly)