Lecture 8
9.25.13

Theorem Let $\mathbb{F}_q$ be a finite field where $q = p^\alpha$ for some prime p, $\alpha \geq 1$. The number of k-dim subspaces of $\mathbb{F}_q^n$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Pf For k-subsets, we counted the ordered k-subsets in 2 ways. We use a similar trick. Let

$G(n,k) = \#$ k-subspaces of $\mathbb{F}_q^n$

$N(n,k) = \#$ linearly independent k-tuples $(v_1, \ldots, v_k)$ in $\mathbb{F}_q^n$

To count $N(n,k)$:

1. • choose $v_1 \neq 0$ $\quad q^n - 1$ choices
   • choose $v_2 \notin \text{span}(v_1)$ $\quad q^n - q$ choices
   • choose $v_3 \notin \text{span}(v_1, v_2)$ $\quad q^n - q^2$ choices
   $\vdots$

So $N(n,k) = (q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$

2. • choose $\text{span}(v_1, \ldots, v_k) = V$ $\quad G(n,k)$ choices
   • choose $v_1 \in V \setminus \{0\}$ $\quad q^k - 1$
   • choose $v_2 \in V \setminus \text{span}(v_1)$ $\quad q^k - q$
   $\vdots$

So $N(n,k) = G(n,k)(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$

Hence

$$G(n,k) = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \blacksquare$$

Another way of counting this:

Given a k-subspace V of $\mathbb{F}_q^n$, we can choose any $k \times n$ matrix A such that $V = \text{rowspace}(A)$. Through elementary row operations, we can get A uniquely to row-reduced echelon form:

$$V = \text{rowspace}\ \boxed{\ A\ } = \text{rowspace} \begin{bmatrix} 0 & 1 & * & * & 0 & * & 0 & 0 \\ & & & & 1 & * & 0 & 0 \\ & & & & & & 1 & 0 \end{bmatrix}$$

Check: this is a bijection

$$(k\text{-subspaces of } \mathbb{F}_q^n) \longleftrightarrow \left(\begin{array}{c}\text{row-reduced echelon form} \\ k \times n \text{ matrices over } \mathbb{F}_q\end{array}\right)$$

So we can just count these matrices, via

$$\begin{array}{c c} & n \\ k & \begin{bmatrix} 1 & * & * & 0 & * & 0 & 0 \\ & & & 1 & * & 0 & 0 \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix} \end{array} \longrightarrow \begin{array}{c c} & n-k \\ k & \begin{bmatrix} * & * & * \\ & & * \\ & & \\ & & \end{bmatrix} \end{array} \longrightarrow \begin{array}{l} \text{"partition" } (3,1) \\ \text{fitting inside} \\ k \times (n-k) \text{ rectangle} \end{array}$$

$\uparrow$ q^4 of these matrices

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_a)$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_a$. If $n = \lambda_1 + \cdots + \lambda_a$ we say λ is a partition of n, and write $\lambda \vdash n$.

Thm $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{m \geq 0} p(k, n-k, m)\, q^m$ where

$p(k, n-k, m) = \#$ of partitions of m with k parts which are $\leq n-k$.

Partitions

Let $p(n)$ = # of partitions of n

$p_{\le k}(n)$ = # of partitions of n with parts $\le k$

Prop
$$\sum_{n\ge 0} p_{\le k}(n)\,x^n = \prod_{i=1}^{k} \frac{1}{1-x^i}$$

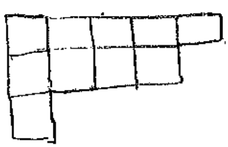
Pf $\quad RHS = \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot \cdots \frac{1}{1-x^k}$

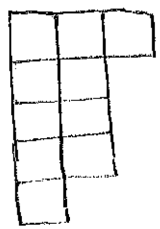
$$= (1+x^1+x^{1+1}+\dots)(1+x^2+x^{2+2}+\dots)\cdots(1+x^k+x^{k+k}+\dots)$$

so the coeff of x^n is the number of ways of writing $n=(1+\dots+1)+(2+\dots+2)+\dots+(k+\dots+k)$ ∎

Prop $\displaystyle\sum_{n\ge 0} p(n)\,x^n = \prod_{i\ge 1}\frac{1}{1-x^i}$

The Ferrers diagram of, e.g. $\lambda=(5,4,1)$ is

$\lambda =$  $=(5,4,1)$

$\longrightarrow \lambda' =$  $=(3,2,2,2,1)$ is the conjugate of λ

The length is $\ell(\lambda)=3$.

Prop There are $p_{\le k}(n)$ partitions of n with $\le k$ parts

Pf λ has $\le k$ parts $\iff \lambda'$ has parts $\le k$.

Prop The number of partitions of n into distinct parts equals the number of partitions of n into odd parts

Pf ① GF:

$$\sum_{n\ge 0} p_{dist}(n)\,x^n = (1+x)(1+x^2)(1+x^3)\cdots \qquad (1)$$

$$\sum_{n\ge 0} p_{odd}(n)\,x^n = \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots \qquad (2)$$

and

$$(1) = \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdots = \prod_{n\ge 0}\frac{1-x^{2n}}{1-x^n} = \prod_{n\ge 0}\frac{1}{1-x^{2n+1}} = (2) \quad ∎$$

② Bij:

Given a partition into distinct parts:

$$111 = 4+6+12+16+20+21+32$$

split each part as $2^x\cdot$odd:

$$111 = 4\cdot1+2\cdot3+4\cdot3+16\cdot1+4\cdot5+1\cdot21+32\cdot1$$

group them by their odd parts:

$$111 = (4+16+32)\cdot1+(2+4)\cdot3+4\cdot5+1\cdot21$$

and let them give repetitions of the odd part:

$$111 = \underbrace{1+1+\dots+1}_{4+16+32=52\text{ times}} + \underbrace{3+\dots+3}_{2+4=6\text{ times}} + \underbrace{5+\dots+5}_{4\text{ times}} + \underbrace{21}_{1\text{ time}}$$

The inverse transformation is: write 52, 6, 4, 1 in binary, and reverse the steps. ∎