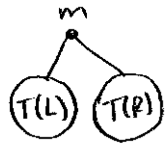
Tree representations of permutations

Lecture 7
9.23.13

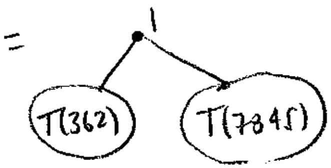
① Represent a perm. π by a tree $T(\pi)$ recursively:

$T(\emptyset) = \emptyset$

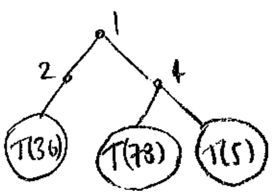
$T(L\, m\, R) =$ 

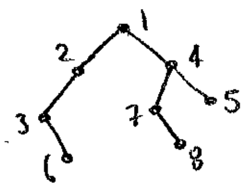
(m: min elt)

Ex:

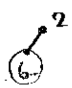
$T(36217845) =$ 

$=$

$=$ 

$= \cdots =$ 

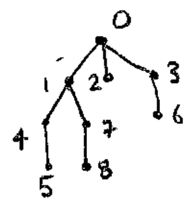
- This is a bijection with increasing binary trees: trees where each vertex may have a L and an R child, and the labels $1,2,\ldots,n$ increase down the tree. (T^{-1} is clear.)
- Stats: Each (descent in w) gives (vx in $T(w)$ with left child)

62 

Prop There are $n!$ binary increasing trees
$A(n,k)$ of them have k vertices with a left child.

② Represent a perm. π by an unordered tree $T'(\pi)$ where

$$\text{parent}(i) = \begin{cases} j & \text{if } j \text{ is the rightmost \# to the left of } i \text{ with } j<i \\ 0 & \text{if none} \end{cases}$$

Ex: 36217845 $\xrightarrow{T'}$ 

parents: 03001714

- This is a bijection with unordered increasing trees on $[0,n]$: trees where the children of a vertex have no specified order, and labels $0,1,\ldots,n$ increase.

Inverse: exercise.

- Stats: - The children of the root 0 correspond to the "anti-records" (L-to-R minima) of π
  - The "leafs" of the tree correspond to the descents of π.

Prop There are $n!$ unordered increasing trees on $[0,n]$.

- $c(n,k)$ of them have k children of the root
- $A(n,k)$ of them have k leaves.

An application: counting permutations by descent set

Let a, b be non-commutative and let the "descent word" of $w_1 \dots w_n$ be $e_1 \dots e_{n-1}$ where

$$e_i = \begin{cases} a & i \text{ not a descent} \\ b & i \text{ is a descent} \end{cases}$$

Let the <u>ab-index</u> of S_n be

$$\Psi_n = \sum_{w \in S_n} (\text{descent word of } w)$$

Ex: $\Psi_3 = \underset{123}{aa} + \underset{132}{ab} + \underset{213}{ba} + \underset{231}{ab} + \underset{312}{ba} + \underset{321}{bb}$

$$= aa + 2ab + 2ba + bb = c^2 + d \qquad \begin{matrix} c = a+b \\ d = ab+ba \end{matrix}$$

$\Psi_4 = c^2 + 2cd + 2dc$

> <u>Thm</u> The ab-index $\Psi_n(a,b)$ equals a non-comm. p $\Phi_n(c,d)$ in $c = a+b$ $d = ab+ba$

A more economical/magical way of storing the numbers $\beta_n(S) = \#$ perms of n with descent set S (2^{n-1} vs Fibonacci terms)

Note. This is a special case of a much more general phenomenon we will discuss later, which holds, e.g., for any polytope.

More on q-binomial/multinomial coeffs

Recall $[n]_q = 1 + q + q^2 + \dots + q^{n-1}$.

$[n]_q! = [1] \cdot [2] \cdot \dots \cdot [n]$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]!\,[n-k]!} \qquad \begin{bmatrix} n \\ a_1 \dots a_k \end{bmatrix}_q = \frac{[n]!}{[a_1]! \dots [a_k]!}$$

Easy: $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$

So: $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is polynomial in q (very unclear from the def!)

$$\begin{bmatrix} n \\ a_1 \dots a_k \end{bmatrix} = \begin{bmatrix} n \\ a_1 \end{bmatrix} \begin{bmatrix} n-a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} n-a_1-a_2 \\ a_3 \end{bmatrix} \dots \begin{bmatrix} n-a_1-\dots-a_{k-1} \\ a_k \end{bmatrix}$$

> <u>Prop</u> Let $M = \{1^{a_1}, 2^{a_2}, \dots, m^{a_m}\}$. Then
>
> $$\sum_{w \text{ perm. of } M} q^{\mathrm{inv}(w)} = \begin{bmatrix} n \\ a_1 \dots a_m \end{bmatrix}$$

<u>Pf</u> Consider the bijection

$$\varphi : S_M \times S_{a_1} \times \dots \times S_{a_m} \longrightarrow S_n$$

$$(132121322, 312, 1324, 21) \mapsto 394162857$$

Note

$\mathrm{inv}(w_0) + \mathrm{inv}(w_1) + \dots + \mathrm{inv}(w_m) = \mathrm{inv}(w)$

so

$$\left(\sum_{w \in S_M} q^{\mathrm{inv}(w_0)} \right) \prod_{i=1}^{m} \left(\sum_{w \in S_{a_i}} q^{\mathrm{inv}(w_i)} \right) = \sum_{w \in S_n} q^{\mathrm{inv}(w)} \qquad \blacksquare$$