Pattern Avoidance

Lecture 6
9.16.13

Let $u = u_1 \dots u_k \in S_k$

$v = v_1 \dots v_n \in S_n \qquad k < n$

We say v <u>avoids</u> the pattern u if there do not exist $i_1 < \dots < i_k$ such that $u_1, \dots, u_k$ are in the same relative order as $v_{i_1}, \dots, v_{i_k}$:

$$u_a < u_b \iff v_{i_a} < v_{i_b}$$

Ex: $\underline{5}3\underline{6}41\underline{2}$ contains 231

341625 avoids 321

Theorem The number of 321-avoiding permutations of $[n]$ is the <u>Catalan number</u>

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

The Catalan numbers appear all over combinatorics. (See Exercise 6.19 for hundreds of interpretations)

Def Let the <u>Catalan number</u> C_n be the number of paths from $(0,0)$ to $(2n,0)$ using steps NE ↗ and SE ↘ ($(1,1)$ and $(1,-1)$) which do not go below the x-axis. "<u>Dyck paths</u>"

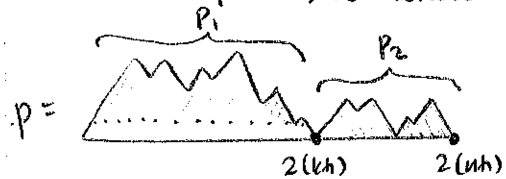
$C_3 = 5$:

Prop $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ for $n \geq 0$

Pf Let $P_n = \{\text{Dyck paths of length } 2n\}$

Given a path $p \in P_{n+1}$, say it first returns to x-axis at $(2(k+1), 0)$ for the first time:

Let $f(p) = (P_1, P_2)$ where



$P_1 \in P_k$

$P_2 \in P_{n-k}$

This is a bijection

$$f: P_{n+1} \to \bigcup_{k=0}^{n} P_k \times P_{n-k} \qquad \blacksquare$$

Thm $C_n = \frac{1}{n+1}\binom{2n}{n}$

Pf Let $C(x) = C_0 + C_1 x + C_2 x^2 + \dots$ Then

$$C(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$$

$$x\,C(x)^2 = \sum_{n=1}^{\infty} C_{n+1} x^{n+1} + x = C(x) - 1$$

$\left((1 + x + 2x^2 + 5x^3 + \dots)^2 = 1 + 2x + 5x^2 + \dots \right)$

$$4x^2 C(x)^2 = 4x\,C(x) - 4x$$

$$[1-2x\,C(x)]^2 = 1-4x$$

$$1-2x\,C(x) = (1-4x)^{1/2} = \sum_{k=0}^{\infty}\binom{1/2}{k}(-4x)^k$$

Equating coeffs of x^{n+1}

$$-2C_n = \binom{1/2}{n+1}(-4)^{n+1}$$

$$= \frac{(1/2)(1/2-1)(1/2-2)\cdots(1/2-n)}{(n+1)!}(-1)^{n+1}4^{n+1}$$

$$= \frac{(-1)\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}(n+1)!}\cdot 4^{n+1}$$

$$2C_n = 1\cdot 3\cdot 5\cdots(2n-1)\cdot\left(\frac{2\cdot 4\cdot 6\cdots 2n}{2^n n!}\right)\cdot\frac{2^{n+1}}{(n+1)!}$$

$$= \frac{(2n)!}{n!\,(n+1)!}\cdot 2$$

$$C_n = \frac{1}{n+1}\cdot\binom{2n}{n} \qquad \blacksquare$$

Two combinatorial proofs:

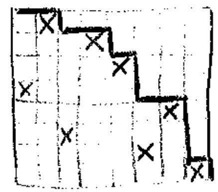
- Out of the $\binom{2n}{n}$ paths ↗ ↘ from $(0,0)$ to $(2n,0)$, $\frac{1}{n+1}$-th of them are Dyck
- $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$
  ↑ non-Dyck paths

We need a bijection from some permutations to Dyck paths.

↑ "on a line"

↑ "on a grid"

Idea: represent perms. on a grid:



$= 24517368$

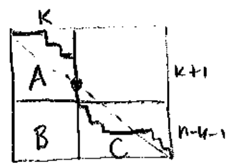
How about

(perms. of S_n) $\xrightarrow{P}$ (paths on a grid)

$w \longmapsto$ lowest path from NW to SE which is above the perm. matrix of w. $= P(w)$

Claim: $P(w)$ is Dyck.

Pf: if not, say it crosses at k:



A, B, C = # of Xs in each region

Then $A+B \le k$ (k columns)

$B+C \le n-k-1$ ($n-k-1$ rows)

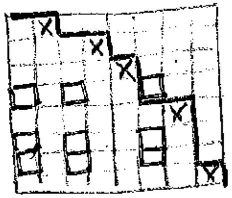
$A+2B+C \le n-1$

But $A+B+C = n$. $\blacksquare$

Claim: P is a bijection

$$\begin{pmatrix}\text{321-avoiding}\\ \text{perms of } [n]\end{pmatrix} \longrightarrow \begin{pmatrix}\text{Dyck paths}\\ \text{of length } 2n\end{pmatrix}$$

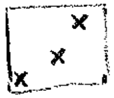
Pf We construct the inverse:



Given P, put Xs at each NW corner of the path.

The other k Xs need to go in the remaining $k \times k$ grid.

We need to avoid 321, i.e.

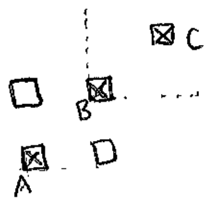


One way to do it: fill

Out of any 3 Xs, two have the same color ⇒ are

Clearly get a permutation.

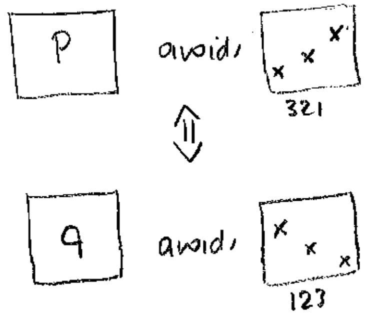
This is the only way. If I do:



either B is above P, or there is a C NE of it, creating a 321 subpattern

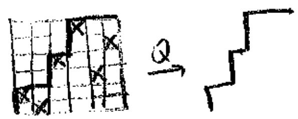
So $\boxed{\text{there are } C_n \text{ permutations of } [n] \text{ avoiding } 321.}$

Note:



So $\boxed{\text{there are } C_n \text{ permutations of } [n] \text{ avoiding } 123}$

I could have also used the map



HW: • Q gives a bijection

$$\begin{pmatrix}\text{132-avoiding}\\ \text{permuts of } [n]\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{Dyck paths}\\ \text{of length } 2n\end{pmatrix}$$

• There are
- C_n 123-avoiding perms of $[n]$
- C_n 132-avoiding
- C_n 213-avoiding
- C_n 231-avoiding
- C_n 312-avoiding
- C_n 321-avoiding

Note: This doesn't generalize to longer patterns.

This is the beginning of <u>pattern avoidance</u>.