Descents

Lecture 5
9.13.13

A *descent* of $w = w_1 \cdots w_n$ is an i with $w_i > w_{i+1}$

Ex: $\mathrm{Des}(1\,5\,2\,8\,4\,3\,6\,7) = \{2,4,5\}$

$\mathrm{des}(15284367) = 3$

The *Eulerian number*

$A(n,k) = \#$ perms. of $[n]$ with $k-1$ descents

The *Eulerian polynomial* is

$$A_n(x) = \sum_{w \in S_n} x^{1+\mathrm{des}(w)} = \sum_{k=1}^{n} A(n,k)\, x^k$$

Ex:

$$A_3(x) = \underset{123}{x} + \underset{132}{x^2} + \underset{213}{x^2} + \underset{231}{x^2} + \underset{312}{x^2} + \underset{321}{x^3} = x + 4x^2 + x^3$$

$A_4(x) = x + 11x^2 + 11x^3 + x^4$ (symmetric)

Claim: $\boxed{A(n,k) = A(n, n+1-k)}$

Pf: Exercise.

Prop $A(n,k) = k\,A(n-1,k) + (n+1-k)\,A(n-1,k-1)$

Pf.

Consider $\varphi : \underbrace{S_{n,k}}_{\text{perms w } k-1 \text{ descents}} \to S_{n-1}$

$\varphi(w_1 \ldots w_i\, n\, w_{i+1} \cdots w_{n-1}) = w_1 \cdots w_i w_{i+1} \cdots w_{n-1}$

If $\left(\begin{matrix} w_i > w_{i+1} \\ i = n-1 \end{matrix}\right)$ $\varphi(w)$ has $k-1$ descents

If $\left(\begin{matrix} w_i < w_{i+1} \\ i = 0 \end{matrix}\right)$ $\varphi(w)$ has $k-2$ descents

So $\mathrm{Im}\,\varphi = S_{n-1,k} \cup S_{n-1,k-1}$.

For $w' \in S_{n-1,k}$, $|\varphi^{-1}(w')| = k$ (insert n between a descent, or at the end)

For $w' \in S_{n-1,k-1}$, $|\varphi^{-1}(w')| = n+1-k$ (insert n between a non-descent, or at beginning)

So

$$A(n,k) = \sum_{w \in \mathrm{Im}\,\varphi} |\varphi^{-1}(w)| = k\,A(n-1,k) + (n+1-k)\,A(n-1,k-1) \quad \blacksquare$$

$$\boxed{\textbf{Prop} \quad \sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}} \quad \leftarrow \text{Euler}$$

Pf Easy induction - apply $x\frac{d}{dx}$ to both sides. $\blacksquare$

Recall $\wedge$:

$\hat{w} = 1\,\underline{5}\,2\,\underline{8}\,\underline{4}\,3\,6\,7 \xleftarrow{\wedge} (1)(52)(84367) = w$

no descent at 36 in $\hat{w}$ $\Leftrightarrow$ $3<6$ $\Leftrightarrow$ $w(3)>3$

A <u>weak excedance</u> (excedance) of $w \in S_n$ is an i with $w(i) \geq i$. ($w(i)>i$)

> **Claim:** w has k weak excedances $\Leftrightarrow$ $\hat{w}$ has $n-k$ descents

$w = (a_1 a_2 \cdots a_{i_1})(a_{i_1+1} \cdots a_{i_2}) \cdots (a_{i_{m-1}+1} \cdots a_{i_m})$

$\hat{w} = a_1 a_2 \ldots a_{i_1} a_{i_1+1} \cdots a_{i_2} \cdots a_{i_{m-1}+1} \cdots a_{i_m}$

a_j weak excedance of $w \Leftrightarrow j = i_r$ or $a_{j+1} \geq a_j$ (since each cycle starts with its max)

$\Leftrightarrow \hat{w}(j+1) > \hat{w}(j)$

$\Leftrightarrow j$ is not a descent. ∎

> **Cor** • There are $A(n,k)$ perms of $[n]$ with k weak exc.
> • " " $A(n,k)$ " " $k-1$ excedances.

Pf a) From above and $A(n,k) = A(n, n-1-k)$

b) Exercise. ∎

Major Index

The <u>major index</u> is

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i$$

> <u>Theorem</u> inv and maj are equidistributed:
> $$\sum q^{\mathrm{inv}(w)} = \sum q^{\mathrm{maj}(w)} = [n]!_q$$

<u>Pf</u> Give a bijection $\varphi: S_n \to S_n$ which sends maj to inv:

$$\mathrm{maj}(w) = \mathrm{inv}(\varphi(w))$$

Given $w = w_1 w_2 \ldots w_n$ define words $\gamma_1, \gamma_2, \ldots, \gamma_n$:

① $\gamma_1 = w_1$

② If I have γ_i:

- split it: if $w_i > w_{i+1}$, split after each $\# > w_{i+1}$
  if $w_i < w_{i+1}$, split after each $\# < w_{i+1}$
- rotate each resulting compartment 1 to the right
- add w_{i+1} at the end, to get γ_{i+1}

③ Let $\varphi(w) = \gamma_n$.

Ex $w = 683941725$

change inv	unsplit		split
	$\gamma_1 = 6$	$6 < 8$	6
$+0$			
	$\gamma_2 = 68$	$8 > 3$	$6\|8$
$+2$			
	$\gamma_3 = 683$	$3 < 9$	$6\|8\|3$
$+0$			
	$\gamma_4 = 6839$	$9 > 4$	$6\|8\|39$
$+4$			
	$\gamma_5 = 68934$	$4 > 1$	$6\|8\|9\|3\|4$
$+5$			
	$\gamma_6 = 689341$	$1 < 7$	$6\|893\|4\|1$
$+0$			
	$\gamma_7 = 6389417$	$7 > 2$	$6\|3\|8\|9\|4\|1\|7$
$+7$			
	$\gamma_8 = 63894712$	$2 < 5$	$63\|894\|71\|2$
$+0$			
	$\gamma_9 = 364891725$		

<u>Claim</u>: $\mathrm{inv}(\gamma_k) = \mathrm{maj}(w_1 w_2 \dots w_k)$

Pf by induction. Assume true for k.

Case 1: <u>$w_k < w_{k+1}$</u>

Then $\mathrm{maj}(w_1 \dots w_{k+1}) = \mathrm{maj}(w_1 \dots w_k)$

Need $\mathrm{inv}(\gamma_{k+1}) = \mathrm{inv}(\gamma_k)$

$$\gamma_k = 6\ 8\ 9\ 3\ 4\ 1 \xrightarrow{<7} 6 | 8 9 3 | 4 | 1$$

lose

$$\downarrow$$

$$6 | 3 8 9 | 4 | 1 | 7$$

win

Each compartment goes

$$|c_1\ c_2 \cdots c_i| \qquad \begin{array}{c} w_{k+1} \\ \| \\ c_i < 7 < c_1, c_2, \dots, c_{i-1} \end{array}$$

$$\downarrow$$

$$|c_i\ c_1 \cdots c_{i-1}|$$

Which loses $i-1$ inversions $(c_1, c_i), \dots, (c_{i-1}, c_i)$

The 7 at the end introduces $i-1$ inversions

$$(c_1, 7), \dots, (c_{i-1}, 7)$$

So the total # of inversions doesn't change.

Case 2: <u>$w_k > w_{k+1}$</u>

Then $\mathrm{maj}(w_1 \dots w_{k+1}) = \mathrm{maj}(w_1 \dots w_k) + k$

Need $\mathrm{inv}(\gamma_{k+1}) = \mathrm{inv}(\gamma_k) + k$. (Similar, see book)

So $\boxed{\mathrm{inv}(\varphi(w)) = \mathrm{maj}(w).}$

<u>Claim</u>: φ is a bijection

Pf: Construct φ^{-1}. (Exercise, see book.) ∎

<u>Remark</u> In fact $(\mathrm{inv}, \mathrm{maj})$ are symmetrically distributed:

$$\boxed{\sum_{\pi \in S_n} x^{\mathrm{inv}(\pi)} y^{\mathrm{maj}(\pi)} = \sum_{\pi \in S_n} x^{\mathrm{maj}(\pi)} y^{\mathrm{inv}(\pi)}}$$