Permutations, revisited

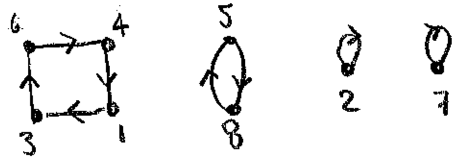
Lecture 4
9.11.13

Counting the $n!$ perms. of $[n]$ was easy.

We now want more refined counts, in terms of various "statistics", e.g., number of cycles, inversions, descents, etc.

Recall various notations for a permutation of $[n]$:

- 2-line: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 6 & 1 & 8 & 4 & 7 & 5 \end{pmatrix}$
- 1-line: 32618475
- cycle: $(1364)(2)(58)(7)$
- digraph:



The cycle notation is not unique. It is useful to choose a

Standard representation:
- In each cycle, largest # is first
- cycles listed by increasing order of largest #

$(1364)(2)(58)(7) = (2)(6413)(7)(85)$

standard

Cycles

$c(n,k) = $ # of perms. of $[n]$ with k cycles

Let $s(n,k) = (-1)^{n-k} c(n,k) =$ "Stirling # of first kind"

Prop $\boxed{c(n,k) = (n-1)\,c(n-1,k) + c(n-1,k-1)}$ (compare w/ $\binom{n}{k}$)

Pf.

Split the perms. into two types:

1. n is in its own cycle; i.e., $\pi(n) = n$.
   Deleting n gives a perm of $[n-1]$ with $k-1$ cycles so there are $c(n-1,k-1)$ of this type.
2. n is not in its own cycle
   Deleting n from the cycle notation gives a perm of $[n-1]$ with k cycles. That n could have followed any of $1,2,\dots,n-1$, so this is an $(n-1)$-to-1 mapping $\Rightarrow (n-1)c(n-1,k)$ of this type. ∎

Prop $\boxed{x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} c(n,k)\,x^k}$ (compare w/ $(x+1)^n$)

Pf Several, see book.

Easiest by induction.

$$[x(x+1)\cdots(x+n-2)](x+n-1) = \left[\sum_{k=0}^{n-1} c(n-1,k)\,x^k\right](x+n-1)$$

$$= \sum_{k=0}^{n} \big((n-1)c(n-1,k) + c(n-1,k-1)\big)\,x^k \; ∎$$

The *type* of $w \in S_n$ is $\text{type}(w) = (c_1, \dots, c_n)$ where w has c_i cycles of length i.

Prop There are $\dfrac{n!}{1^{c_1} c_1! \, 2^{c_2} c_2! \cdots n^{c_n} c_n!}$ perms of type $(c_1 \dots c_n)$

Pf To make such a perm., fill in the blanks:

$$\underbrace{(\_)\cdots(\_)}_{c_1}\underbrace{(\_\,\_)\cdots(\_\,\_)}_{c_2}\underbrace{(\_\,\_\,\_)\cdots}_{c_3}\quad\cdots$$

This gives $\Phi : S_n \to S_n^c$ surjective.
what we are counting

Ex: $c = (3,2,1,0,0,0,0,0,0,0)$

$4\,8\,5\,7\,1\,6\,2\,3\,9\,10 \mapsto (4)(8)(5)(71)(62)(3\,9\,10) = w$

Preimages of w:

- permute cycles of length 1: $c_1!$ $\quad (4), (8), (5)$
- " " " " 2: $c_2!$ $\quad (71), (62)$
- rotate each cycle of length 2: 2^{c_2}

⋮

So this is a $(c_1! 1^{c_1} c_2! 2^{c_2} c_3! 3^{c_3} \dots)$-to-1 mapping. ∎

Let $t^{\text{type}(w)} = t_1^{c_1} \cdots t_n^{c_n}$ and let the *cycle indicator* of S_n be:

$$Z_n = Z_n(t_1 \dots t_n) = \frac{1}{n!} \sum_{w \in S_n} t^{\text{type}(w)}$$

Ex: $Z_3 = \frac{1}{6}\Big(\underset{(1)(2)(3)}{t_1^3} + \underset{(a)(bc)}{3 t_1 t_2} + \underset{(abc)}{2 t_3} \Big)$

Cor $\displaystyle \sum_{n \geq 0} Z_n x^n = \exp\left(t_1 x + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \dots \right)$

Pf "Just plug in". (See book.) ∎

The fundamental bijection:

Let $\wedge : S_n \longrightarrow S_n$ be

$w = (2)(6413)(7)(85) \mapsto 2\,6\,4\,1\,3\,7\,8\,5 = \hat{w}$

(standard cycle rep) $\quad$ (1-line notation)

Prop $\wedge$ is a bijection.
If w has k cycles, $\hat{w}$ has k "records"
L-to-R maxima

Pf To get w from $\hat{w}$, start a new cycle at each record. ∎

Cor. Enum S_n by records

Inversions:

An *inversion* of $w = w_1 w_2 \cdots w_n$ is a pair (w_i, w_j) such that $i<j$, $w_i > w_j$. Let $\text{inv}(w)$ be the number of inversions

Ex: $\text{inv}(2\,4\,1\,6\,3\,5) = 5$

Goal: Enum. of S_n by inversions.

The <u>inversion table</u> of $w \in S_n$ is $(a_1 \ldots a_n)$

where $a_i = \#\{j : (j,i) \text{ is an inversion of } w\}$

Note: • $\operatorname{inv}(w) = a_1 + \cdots + a_n$ • $a_i \le n-i$

Ex: $2\,4\,1\,6\,3\,5 \mapsto (2,0,2,0,1,0) = I(w)$

$\boxed{\textbf{Prop}\ \text{The map } w \mapsto I(w) \text{ is a bijection } S_n \to \{(a_1,\ldots,a_n) : 0 \le a_i \le n-i\}}$

Pf. It suffices to construct the inverse map.

Given $(a_1,\ldots,a_n)$, build words $w^n, w^{n-1}, \ldots, w^2, w^1 = w$ where w^i is obtained from w^{i+1} by inserting i so there are a_i $(\le n-i)$ numbers to its left.

$(2,0,2,0,1,0)$:

$w^6 =$	6	$a_6 = 0$
$w^5 =$	65	$a_5 = 1$
$w^4 =$	465	$a_4 = 0$
$w^3 =$	4635	$a_3 = 2$
$w^2 =$	24635	$a_2 = 0$
$w^1 =$	241635	$a_1 = 2$

Note that after w^i, the value of a_i stays put, so $I(w) = (a_1 \ldots a_n)$ ∎

$\boxed{\textbf{Cor.}\ \sum_{w \in S_n} q^{\operatorname{inv}(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})}$

Pf. $$\sum_{w \in S_n} q^{\operatorname{inv}(\pi)} = \sum_{\substack{(a_1 \ldots a_n) \\ 0 \le a_i \le n-i}} q^{a_1 + \cdots + a_n}$$

$$= \left(\sum_{a_1=0}^{n-1} q^{a_1}\right)\left(\sum_{a_2=0}^{n-2} q^{a_2}\right)\cdots \quad ∎$$

<u>Remark</u>: "q-analogs"

- $1+q+\cdots+q^{n-1} = [n]_q$ is the "q-analog" of $n \in \mathbb{N}$
- $(1+q)\cdots(1+\cdots+q^{n-1}) = [n]_q!$ is the "q-analog" of $n!$

A "q-analog" of a combin. object is an object depending on q which "reduces" to the object when $q=1$. Vague but very common, useful.

Very Common: q prime power

$\mathbb{F}_q$ finite field of q elements

- Vector space $\mathbb{F}_q^n$ is the q-analog of $\{1,\ldots,q\}$

$\#\{\emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_{n-1} \subsetneq S_n = [n]\} = n!$

$\#\{0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{F}_q^n\} = [n]_q!$