Prop There are 2^{n-1} compositions of n
$\binom{n-1}{k-1}$ k-comps of n

Pf To choose a comp of n

- write $n = 1+1+1+1+\cdots+1+1$
- delete some of the +s (2^{n-1} choices)
- group consecutive 1s into one part

(Ex: $8 = 1+1+1+1+1+1+1+1 \Rightarrow 8 = 1+3+2+2$)

Similarly for k-comps. ∎

Prop There are $\binom{n+k-1}{k-1}$ weak k-comps of n

Pf $n = a_1 + \cdots + a_k$ is a weak k-comp of n

$$\Updownarrow$$

$n+k = (a_1+1)+\cdots+(a_k+1)$ is a k-comp of $n+k$. ∎

Lecture 3
9.06.13

Multisets

A multiset is a set with possibly repeated elements, like $\{1,2,2,2,4,5,5,7\} = \{1^1, 2^3, 4^1, 5^2, 7^1\}$ ← notation

This multiset has cardinality/size 8.

Let $\left(\!\binom{S}{k}\!\right) = \{$k-multisets on $S\}$. Let $\left(\!\binom{n}{k}\!\right) = \left|\left(\!\binom{S}{k}\!\right)\right|$ for $|S|=n$.

↑ elts. chosen from S

Prop $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

Pf If a k-multiset S on $[n]$ has a_i copies of i ($1\le i\le n$) then $a_1+\cdots+a_n = k$ is a weak n-composition of k, and viceversa. So

$$\left(\!\binom{n}{k}\!\right) = \binom{k+n-1}{n-1} = \binom{n+k-1}{k}$$ ∎

Multisets and GFs

The multivariate GF for multisets on $[n]$ is

$$\sum_{v:[n]\to\mathbb{N}} \prod_{i=1}^{n} x_i^{v(i)} = (1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots(1+x_n+x_n^2+\cdots)$$

Again letting $x_1 = \cdots = x_n = x$,

$$\sum_{v:[n]\to\mathbb{N}} x^{v(1)+\cdots+v(n)} = (1+x+x^2+\cdots)^n$$

$$\sum_{M \text{ multiset on } [n]} x^{|M|} = \left(\frac{1}{1-x}\right)^n$$

$$\sum_{k=0}^{\infty} \left(\!\binom{n}{k}\!\right) x^k = (1-x)^{-n} =: \sum_{k\ge 0} \binom{-n}{k}(-x)^k$$

binomial thm – true for any exponent

So

$$\boxed{\left(\!\binom{n}{k}\!\right) = (-1)^k \binom{-n}{k}}$$

"combin. reciprocity thm"

The <u>multinomial coefficient</u> $\binom{n}{a_1,\dots,a_m}$ is the number of ways of splitting an n-set into an a_1-set, an a_2-set, ..., and an a_m-set in order (where $a_1+\dots+a_m=n$)

For ex. $\binom{n}{k,n-k}=\binom{n}{k}$

Prop The number of permutations of $\{1^{a_1},2^{a_2},\dots,m^{a_m}\}$ is $\binom{n}{a_1,\dots,a_m}$ where $a_1+\dots+a_m=n$

Pf Out of the $a_1+\dots+a_m$ positions — — — — — — —

choose which a_1 of them will hold the 1s

a_2 2s

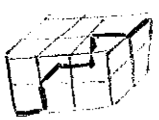
⋮

a_m ms

Prop $\binom{n}{a_1,\dots,a_m}=\frac{n!}{a_1!\cdots a_m!}$

Prop $(x_1+\dots+x_m)^n=\sum_{\substack{a_1+\dots+a_m=n\\ a_i\geq 0}}\binom{n}{a_1\dots a_m}x_1^{a_1}\cdots x_m^{a_m}$

Prop In the m-dim. box of dimensions $a_1\times\dots\times a_m$, there are $\binom{n}{a_1\dots a_m}$ shortest lattice paths from $(0,\dots,0)$ to $(a_1,\dots,a_m)$



$\binom{7}{3,2,2}$

<u>Combinatorial Identities</u>

- $\binom{n}{k}=\binom{n}{n-k}$

  Alg.: Clear

  Comb.: Bijection (k-subsets of $[n]$) $\longleftrightarrow$ ($(n-k)$-subsets of $[n]$)

  $S \mapsto [n]\setminus S$

- $\binom{n+1}{k+1}=\binom{n}{k}+\binom{n}{k+1}$

  Alg.: Easy

  Comb.: Count the $(k+1)$-subsets of $[n+1]$ differently.

  - subsets not containing $n+1$: $\binom{n}{k+1}$
  - subsets containing $n+1$: $\binom{n}{k}$

- $\binom{m+n}{k}=\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$

  Alg: Compare the coeffs of x^k in

  $(1+x)^{m+n}=(1+x)^m(1+x)^n$

  Comb.: Consider a set of m blue, n red elements.

  Count the k-subsets in two ways:

  ⓐ $\binom{m+n}{k}$

  ⓑ If there are i blue elts in the k-subset, there must be $k-i$ red elts. So there are $\binom{m}{i}\binom{n}{k-i}$ possibilities. Now add over all possible i.