Lecture 27
12.12.13

Proof of Dehn-Sommerville relations:

Recall the zeta polynomial

$$Z(L,n) = \zeta^n(\hat{0},\hat{1})$$

$$= \sum_{\hat{0}\le t_1\le \dots \le t_{n-1}\le \hat{1}} \zeta(\hat{0},t_1)\zeta(t_1,t_2)\cdots \zeta(t_{n-1},\hat{1})$$

$$\boxed{Z(L,n) = \#\text{ of multichains of length } n \text{ from } \hat{0} \text{ to } \hat{1}}$$

Now

$$Z(L,-n) = \zeta^{-n}(\hat{0},\hat{1})$$

$$= \mu^n(\hat{0},\hat{1})$$

$$= \sum_{\hat{0}\le t_1\le \dots \le t_{n-1}\le \hat{1}} \mu(\hat{0},t_1)\mu(t_1,t_2)\cdots \mu(t_{n-1},\hat{1})$$

$$= \sum_{\hat{0}\le t_1\le \dots \le t_{n-1}\le \hat{1}} (-1)^{l(\hat{0},t_1)}(-1)^{l(t_1,t_2)}\cdots (-1)^{l(t_{n-1},\hat{1})}$$

$$= (-1)^d(\#\text{ of multichains of length } n \text{ from } \hat{0} \text{ to } \hat{1})$$

$$\boxed{Z(L,-n) = (-1)^d Z(L,n)}$$

Also

$$Z(L,n) = \sum_{m=0}^{n} Z(L-\hat{1},m) \qquad (\text{use } \hat{1} \ n-m \text{ times})$$

So

$$\boxed{Z(L,n) - Z(L,n-1) = Z(L-\hat{1},n)}$$

Also

$$Z(L-\hat{1},n) = \sum_i f_{i-1} Z(B_i, n-1)$$

Let the last face F be (i-1)-dim, then $[\hat{0},F] \cong B_i$

$$= \sum_i f_{i-1}(n-1)^i$$

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So:

$$\sum_i f_{i-1}(n-1)^i = Z(L-\hat{1},n) = Z(L,n) - Z(L,n-1)$$

$$(-1)^d \sum_i f_{i-1}(-n)^i = (-1)^d Z(L-\hat{1},-n+1) = (-1)^d (Z(L,-n) - Z(L,-n+1))$$

Hence

$$\sum_i f_{i-1}(n-1)^i = (-1)^d \sum_i f_{i-1}(-n)^i$$

$$(n-1)^d \sum_i f_{i-1}\left(\frac{1}{n-1}\right)^{d-i} = (-1)^d(-n)^d \sum_i f_{i-1}\left(\frac{-1}{n}\right)^{d-i}$$

$$(n-1)^d \sum_k h_k\left(\frac{1}{n-1}+1\right)^{d-k} = n^d \sum_k h_k\left(-\frac{1}{n}+1\right)^{d-k}$$

$$\sum_k h_k\left(\frac{n}{n-1}\right)^{d-k} = \left(\frac{n}{n-1}\right)^d \sum_k h_k\left(\frac{n-1}{n}\right)^{d-k}$$

$$h(x) = x^d h(1/x)$$

$$h_i = h_{d-i} \qquad \blacksquare$$

To prove there are no others, need to construct enough polytopes to "span" all other f-vectors.

This characterizes the equalities satisfied by the f-vector of a simplicial polytope. Ineqs?

Remarkably, the celebrated g-theorem characterizes exactly which vectors are f-vectors of simplicial polytopes. (McMullen 70, Billera-Lee 79, Stanley 79)

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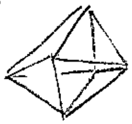
What if we count not only the faces but also their incidences?

The flag f-vector of P is

$$f(S) = \#\text{ of flags of faces } F_1 \subset \cdots \subset F_k \text{ of dimensions } \{a_1 < \cdots < a_k\} = S$$

Ex:



$f(\emptyset)=1$ $\quad f(0)=6$ $\quad f(1)=12$ $\quad f(2)=8$

$f(01)=24$ $\quad f(02)=24$ $\quad f(12)=24$ $\quad f(123)=48$

Encode this in the non-commutative polynomial

$$X_p(a,b) = aaa + 6baa + 12aba + 8aab + 24bba + 24bab + 24abb + 48bbb$$

Let the <u>ab-index</u> of P be

$$\Psi_p(a,b) = X_p(a-b, b)$$

$$= 1aaa + 5baa + 11aba + 7aab + 7bba + 11bab + 5abb + 1bbb$$

and let these coefficients be the <u>flag h-vector</u> of P

Easy: $h(S) = \sum_{T \subseteq S} (-1)^{|S-T|} f(T)$ $\qquad f(S) = \sum_{T \subseteq S} h(T)$

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Are there further relations among these?

For example, $h(S) = h(\overline{S})$!

Note: $\Psi_p(a,b) = (a+b)^3 + 6(a+b)(ab+ba) + 4(ab+ba)(a+b)$

> <u>Theorem</u>. (Margie Bayer, Lou Billera) <u>"cd-index"</u>
>
> For every polytope (or Eulerian poset) there exists a polynomial $\Phi_p(c,d)$ in non-commutative variables c,d such that
>
> $$\Psi_p(a,b) = \Phi_p(a+b, ab+ba)$$

(Also, this determines all the relations among the f(S).)

∘ The $8=2^3$ entries of the flag f-vector of a 3-polytope depends only on the 3 coeffs of

c^3 $\qquad cd$ $\qquad dc$.

Because $\deg c = 1$, $\deg d = 2$, there are F_{n+1} cd-monomials of degree n. So:

> The 2^n entries of the flag f-vector of a polytope are determined completely by $F_{n+1} \approx 1.61^n$ of the entries, and no fewer!