Theorem (Phillip Hall)

Let P be a finite poset, $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$

Let c_i = # of chains $\hat{0} = b_0 < b_1 < \cdots < b_i = \hat{1}$ of length i

Then

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots$$

Pf In the incidence algebra of $\hat{P}$ we have

Recall: $(\zeta-1)^k(x,y)$ = # of k-chains from x to y

$$\mu = \zeta^{-1}$$
$$= (1 + \zeta - 1)^{-1}$$
$$= 1 - (\zeta-1) + (\zeta-1)^2 - \cdots$$

so

$$\mu(\hat{0}, \hat{1}) = 1 - c_1 + c_2 - \cdots \quad \blacksquare$$

This has a topological meaning.

A (abstract) simplicial complex on V is a collection Δ of subsets of V ("faces") such that

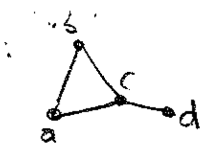
- If $v \in V$ then $\{v\} \in \Delta$
- If $F \in \Delta$ and $G \subseteq F$ then $F \in \Delta$

(Equiv: an order ideal of Boolean lattice B_V containing all singletons.)

Ex: $V = \{a,b,c,d\}$

$\Delta = \{\emptyset, a, b, c, ab, ac, bc, cd\}$

Draw:



(also a geom/top object)

Let $f_i(\Delta)$ = # of faces of dim i (size $i+1$)

Let $\tilde{\chi}(\Delta) = -1 + f_0 - f_1 + f_2 - \cdots$ be the reduced Euler characteristic of Δ.

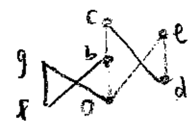
(the "-1" at the beginning for $\Delta \neq \emptyset$)

The order complex of a poset P is

$$\Delta(P) = \{\text{chains of } P\}$$

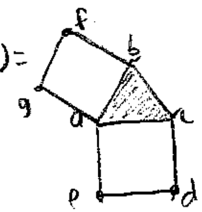
Theorem

$$\boxed{\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))}$$



Compute $\mu(\hat{0}, \hat{1})$

$\Delta(P) =$



A good reason for combinatorialists to learn topology!

Some basic properties:

- $\tilde{\chi}(\Delta)$ only depends on the topological space $|\Delta|$.

$$\tilde{\chi}(\Delta) = \beta_0 - \beta_1 + \beta_2 - \dots$$

$$\beta_i = \operatorname{rank} \underset{\substack{\uparrow \\ \text{"homology gp"}}}{\tilde{H}_i(\Delta;\mathbb{Z})}$$ # of i-dim holes

- $\tilde{\chi}$ doesn't change under homeomorphism, homotopy

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- $\underset{\substack{\uparrow \\ \text{n-ball}}}{\tilde{\chi}(B^n)} = 0$ $\underset{\substack{\uparrow \\ \text{n-sphere}}}{\tilde{\chi}(S^n)} = (-1)^n$

- $\tilde{\chi}$ (bouquet of k d-dim spheres) $= k(-1)^d$

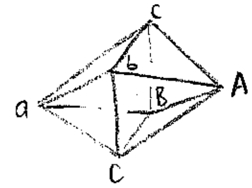
So it is very useful to study the topological properties of simplicial complexes arising in combinatorics, and there are many tools to do this. Search: "Topological Combinatorics"

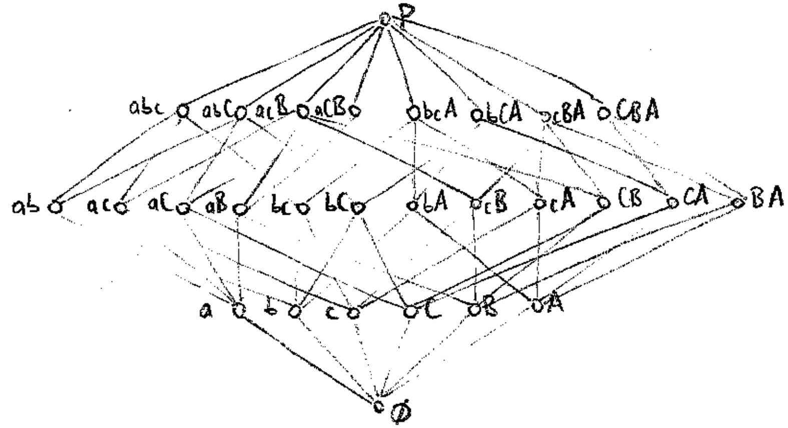
"Poset topology" (Michelle Wachs)

Face lattices of polytopes

Let P be a polytope

Let $L(P)$ be the poset of faces of P, ordered by inclusion (including the "empty face")

Ex. $P =$ 



Theorem (Euler)

If P is a d-dim polytope and $f_i = $ # of i-dim faces then

$$-1 + f_0 - f_1 + \dots \pm f_{d-1} \mp f_d = 0$$

Corollary

$$\mu(\hat{0}, F) = (-1)^{\dim F}$$

$$\mu(F, G) = (-1)^{\dim G - \dim F}$$

- L is a lattice

$F \wedge G = F \cap G$ $F \vee G = \bigwedge_{\substack{H \geq F \\ H \geq G}} H$

- If $L = L(P)$, $L^{op} = L(P^\Delta)$

$L^{op} = L$ upside down

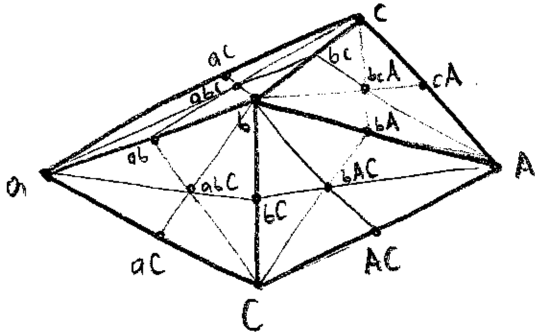
P^Δ = "polar polytope"

- $[\hat{0}, F] = L(F)$ • $[F, \hat{1}] = L(F^\Delta)$

- $[F, G]$ is the face lattice of a polytope for any $F \leq G$.

One nice explanation:

If Γ is a polyhedral complex (polytopes glued together face to face) and P is the face poset of Γ then $\Delta(P(\Gamma))$ = barycentric subdiv of $\Gamma \approx |\Gamma|$



Let P be a simplicial polytope, so every proper face is a simplex. Let f_i = # of i-dim faces.

Euler: $f_0 - f_1 + f_2 - \cdots + (-1)^d f_d = \begin{cases} 0 & d \text{ even} \\ 2 & d \text{ odd} \end{cases}$

But these aren't the only relations!

Ex ($d=6$)

$$f_0 - f_1 + f_2 + f_3 + f_4 = 2$$
$$2f_1 - 3f_2 + 4f_3 - 5f_4 = 0$$
$$2f_3 - 5f_4 = 0$$

Let the h-vector of P be given by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{k=0}^{d} h_k x^{d-k}$$

Ex  f-vector: $(1, 6, 12, 8)$

$$(x-1)^3 + 6(x-1)^2 + 12(x-1) + 8 = x^3 + 3x^2 + 3x + 1$$

h-vector: $(1, 3, 3, 1)$

Theorem (Dehn-Sommerville)

For any simplicial d-polytope,

$$h_i = h_{d-i}$$

Also, any linear relation satisfied by all f-vectors of all simplicial d-polytopes is a consequence of these

So f-vectors of simplicial d-polytopes have $\lceil d/2 \rceil$ degrees of freedom

Stanley's trick to get h-vector from f-vector:

1
1 6
1 5 12 → f-vector
1 4 7 8
1 3 3 1 → h-vector

So $f_0 = 6$ determines $f_1 = 12$ and $f_2 = 8$!