Möbius Inversion Formula

Let P be a poset

Let $f,g : P \to \mathbb{R}$ be such that

$$g(t) = \sum_{s \le t} f(s) \qquad \text{for all } t \in P$$

Then

$$f(t) = \sum_{s \le t} \mu(s,t)\, g(s)$$

1st Pf For any t,

$$\sum_{s \le t} \mu(s,t)\Big(\sum_{r \le s} g(r)\Big) = \sum_r g(r) \sum_{r \le s \le t} \mu(s,t)$$

$$= \sum_r g(r)\, [\zeta \cdot \mu\, (r,t)]$$

$$= \sum_r g(r)\, 1(r,t) = g(t) \quad \blacksquare$$

2nd Pf $g = \zeta f \iff f = \mu g$

For details, see book $\blacksquare$

Over the next few classes we will discuss Möbius functions and inversion more slowly + combinatorially.

They are very important

Some Möbius examples

Lemma

If P, Q are posets, then

$$\mu_{P\times Q}((p,q),(p',q')) = \mu_P(p,p')\,\mu_Q(q,q')$$

Pf. We need

$$\sum_{(s,t) \le (p,q) \le (s',t')} \mu_{P\times Q}((s,t),(p,q)) = \sum_{\substack{s \le p \le s' \\ t \le q \le t'}} \mu_P(s,p)\,\mu_Q(t,q)$$

$$= \sum_{s \le p \le s'} \mu_P(s,p) \sum_{t \le q \le t'} \mu_Q(t,q) = 0 \cdot 0 = 0$$

$\blacksquare$

$P = B_n$ (Boolean lattice)

$B_n \cong \mathbf{2}^n \qquad \mathbf{2} : \begin{smallmatrix}1\\0\end{smallmatrix} \qquad \mu_2(0,0) = \mu_2(1,1) = 1 \qquad \mu_2(0,1) = -1$

$$\mu_{B_n}(S,T) = \mu_{B_n}(\underbrace{0100101}_{S}, \underbrace{0110111}_{T}) = (-1)^{|T-S|}$$

Möbius inversion = Inclusion-Exclusion:

$$g(S) = \sum_{T \subseteq S} f(T) \iff f(S) = \sum_{T \subseteq S} (-1)^{|S-T|} g(T)$$

$P = D_n$ (divisors of n)

$D_{60} =$ 

$n = p_1^{\alpha_1}\cdots p_k^{\alpha_k} \longrightarrow D_n \cong (\alpha_1+1)\times\cdots\times(\alpha_k+1)$

$b \mapsto (\beta_1, \ldots, \beta_k) \qquad b = p_1^{\beta_1}\cdots p_k^{\beta_k}$

$$\mu_{D_n}(b,c) = \mu_{D_n}((\beta_1,\ldots,\beta_k),(\gamma_1,\ldots,\gamma_k))$$
$$= \prod_{i=1}^{k} \mu_{\alpha_i+1}(\beta_i,\gamma_i) \qquad \mu_{\alpha+1}(\beta,\gamma) = \begin{cases} 1 & \gamma=\beta \\ -1 & \gamma=\beta+1 \\ 0 & \gamma\geq\beta+2 \end{cases}$$
$$= \begin{cases} 0 & \text{if some } \gamma_i-\beta_i \geq 2 \\ (-1)^t & t = \#\, i \text{ s.t. } \gamma_i-\beta_i = 1 \quad \text{otherwise} \end{cases}$$

$$\mu(c/b) =: \begin{cases} 0 & \text{if some } p^2 \mid c/b \\ (-1)^t & \text{if } c/b \text{ is a product of } t \text{ distinct primes} \end{cases}$$

Möbius Inversion

$$g(n) = \sum_{d\mid n} f(d) \iff f(n) = \sum_{d\mid n} \mu(n/d)\, g(d)$$
$$= \sum_{\{i_1,\ldots,i_t\}\subseteq[k]} (-1)^t\, g\left(\frac{n}{p_{i_1}\cdots p_{i_t}}\right) \qquad (n = p_1\cdots p_k)$$

Two other examples without proof:

L = distributive lattice = $J(P)$

$$\mu_L(I,J) = \begin{cases} (-1)^t & \text{if } [I,J]\cong B_t \text{ is Boolean} \\ & (J-I \text{ is an antichain in } P) \\ 0 & \text{otherwise} \end{cases}$$

(not too hard)

$P = \Pi_n$ (partition lattice)

Note that intervals:

$$[14\text{-}2\text{-}35\text{-}68\text{-}79,\ 134568\text{-}279] = \Pi_{\{14,35,68\}} \times \Pi_{\{2,79\}}$$
$$= \Pi_3 \times \Pi_2$$

are products of Π_i's, so it is enough to find $\mu_n = \mu_{\Pi_n}(\hat{0},\hat{1})$.

(In ex. above, $\mu([\ ,\ ]) = \mu_3\mu_2$). We have

$$\mu_n = (n-1)!\,(-1)^{n-1}$$

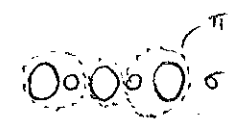
There are several nice proofs, none is very simple.

One: notice that

$$\underbrace{q^n}_{\text{q-colorings of } [n]} = \sum_{\underbrace{\pi\in\Pi_n}_{\text{choose blocks}}} \underbrace{q(q-1)\cdots(q-|\pi|+1)}_{\text{color each block}}$$

Similarly

$$\underbrace{q^{|\sigma|}}_{\text{q-colorings of blocks of } \sigma} = \sum_{\pi\geq\sigma} q(q-1)\cdots(q-|\pi|+1)$$



so Möbius inversion gives

$$q(q-1)\cdots(q-n+1) = \sum_{\pi\in\Pi_n} \mu(\hat{0},\pi)\, q^{|\pi|}.$$ Then compare coeffs of q.

Theorem (Phillip Hall)

Let P be a finite poset, $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$

let c_i = # of chains $\hat{0} = t_0 < t_1 < \cdots < t_i = \hat{1}$ of length i.

Then

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots$$

Pf In the incidence algebra of $\hat{P}$ we have

$$\mu = \zeta^{-1}$$
$$= (1 + \zeta - 1)^{-1}$$
$$= 1 - (\zeta - 1) + (\zeta - 1)^2 - \cdots$$

Recall: $(\zeta-1)^k(x,y)$ = # of k-chains from x to y

so

$$\mu(\hat{0}, \hat{1}) = 1 - c_1 + c_2 - \cdots \quad ■$$

This has a topological meaning.

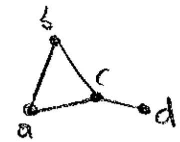
A (abstract) simplicial complex on V is a collection Δ of subsets of V ("faces") such that

- If $v \in V$ then $\{v\} \in \Delta$
- If $F \in \Delta$ and $G \subseteq F$ then $F \in \Delta$

(Equiv: an order ideal of Boolean lattice B_V containing all singletons.)

Ex: $V = \{a, b, c, d\}$

$\Delta = \{\emptyset, a, b, c, ab, ac, bc, cd\}$

Draw: 

Let $f_i(\Delta)$ = # of faces of dim i (size $i+1$)

let $\tilde{\chi}(\Delta) = -1 + f_0 - f_1 + f_2 - \cdots$ be the reduced Euler characteristic of Δ.

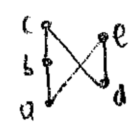
(the "-1" at the beginning for $\Delta \neq \emptyset$)

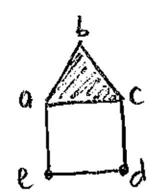
The order complex of a poset P is

$\Delta(P) = \{\text{chains of } P\}$

Theorem

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))$$

Ex: $P =$ 

$\Delta(P) =$ 

A good reason for combinatorialists to learn topology!