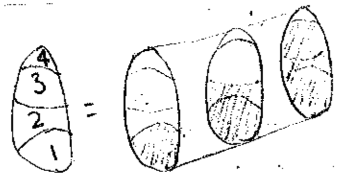
lecture 24
11.21.13

Prop P poset, $m \in \mathbb{N}$. These are equal:

ⓐ # of order-preserving maps $P \to \mathbf{m}$

ⓑ # of multichains $\hat{0} = I_0 \le I_1 \le \dots \le I_m = \hat{1}$ in $J(P)$ of length m

ⓒ $|J(P \times (\mathbf{m-1}))|$

Pf: ⓐ = ⓑ =  = ⓒ

Prop $\left(\begin{matrix}\text{\# of linear extensions}\\ \text{of } P\end{matrix}\right) = \left(\begin{matrix}\text{\# of maximal chains}\\ \text{of } J(P)\end{matrix}\right)$

Let $e(P)$ = # of linear extensions of P.

These are also lattice paths in $\mathbb{N}^w$.

Ex. $e(\mathbf{m}+\mathbf{n}) = \binom{m+n}{m}$

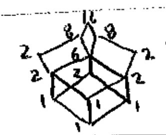
$e(\mathbf{2}\times\mathbf{n}) = C_n = \frac{1}{n+1}\binom{2n}{n}$

Prop If $p_1, \dots, p_k$ are the maxl elts of P,

$$e(P) = e(P-p_1) + \dots + e(P-p_k)$$

Pf # of lin. ext φ where $\varphi(p_i) = n$ $= e(P-p_i)$

Cor $e(P)$ can be computed by applying the "generalized Pascal recurrence" on $J(P)$

Ex $e(M) =$  $= 16$

The zeta polynomial of P is given by

$$Z(P,n) = \text{\# of multichains } t_1 \le t_2 \le \dots \le t_{n-1} \text{ in } P$$

This is indeed a polynomial; if b_i = # of chains $t_1 < \dots < t_{i-1}$ in P,

$$Z(P,n) = \sum_{i \ge 2} b_i \binom{n-2}{i-2}$$

↑ compositions of $n-1$ into $i-1$ parts (a polynomial in n of deg $i-2$)

$\deg Z(P,n) = ht(P)$

The order polynomial of P is given by

$$\Omega_P(m) = \text{\# of order-preserving maps } P \to \mathbf{m} = Z(J(P), m)$$

$\deg \Omega_P(m) = |P|$

leading coeff $= e(P)/|P|!$

There is a nice algebraic approach.

Incidence Algebras

The incidence algebra $I(P)$ of a poset P is the $\mathbb{R}$-algebra of functions

$$\boxed{f : \mathrm{Int}(P) \to \mathbb{R}}$$

($\mathbb{R}$-algebra: ring that is also an $\mathbb{R}$-vector space.)

(where $\mathrm{Int}(P) = \{[x,y] : x \le y \text{ in } P\}$) with multiplication

$$\boxed{fg(x,z) = \sum_{x \le y \le z} f(x,y)\, g(y,z)} \qquad \text{(convolution)}$$

This ring has a multiplicative identity $\boxed{1(x,y) = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}}$

The <u>zeta function</u> of P is

$$\boxed{\zeta(x,y) = 1 \quad \text{for all } x,y \in P}$$

Then

$$\zeta^2(x,y) = \sum_{x \le z \le y} \zeta(x,z)\zeta(z,y) = \sum_{x \le z \le y} 1 = |[x,y]|$$

and

$$\zeta^n(x,y) = \sum_{x \le t_1 \le \cdots \le t_{n-1} \le y} \zeta(x,t_1)\zeta(t_1,t_2)\cdots \zeta(t_{n-1},y)$$

$$= \sum_{x \le t_1 \le \cdots \le t_{n-1} \le y} 1 = \boxed{Z([x,y],n) = \zeta^n(x,y)}$$

Similarly, since

$$(\zeta - 1)(x,y) = \begin{cases} 1 & x < y \\ 0 & x = y \end{cases}$$

we have

$$(\zeta-1)^n(x,y) = \#\text{ of chains } x < t_1 < \cdots < t_{n-1} < y.$$

Prop The following are equivalent for $f \in I(P)$:

- f has a left-inverse
- f has a right-inverse
- f has a two-sided inverse
- $f(s,s) \neq 0$ for all $s \in P$.

Pf See book.

Since $(2-\zeta)(x,y) = \begin{cases} 1 & x = y \\ -1 & x < y \end{cases}$, $(2-\zeta)^{-1}$ exists. $\quad (2 = 2\cdot 1)$

Then

$$\boxed{(2-\zeta)^{-1}(x,y) = \#\text{ of chains from } x \text{ to } y.}$$

Pf:

$$(2-\zeta)^{-1} = (1-(\zeta-1))^{-1} = \underbrace{1}_{\text{0-chain}} + \underbrace{(\zeta-1)}_{\text{1-chain}} + \underbrace{(\zeta-1)^2}_{\text{2-chain}} + \cdots + \underbrace{(\zeta-1)^h}_{h\text{-chain}} \qquad h = h(x,y)$$

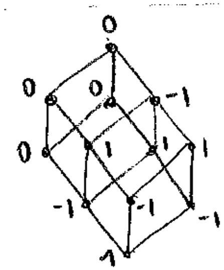
Note: ζ is invertible so let the <u>Möbius function</u> of P be

$$\boxed{\mu = \zeta^{-1}}$$

Equivalently,

$$\boxed{\mu(x,y) = \begin{cases} 1 & x = y \\ -\sum_{x \le z < y} \mu(x,z) & x < y \end{cases}}$$

Ex: D_{84}:



$\mu(\hat{0}, x)$:

$$\boxed{\begin{array}{l}\underline{\text{Möbius Inversion Formula}}\\ \text{Let } P \text{ be a poset}\\ \text{Let } f,g : P \to \mathbb{R} \text{ be such that}\\ \qquad g(t) = \sum_{s \le t} f(s) \qquad \text{for all } t \in P\\ \text{Then}\\ \qquad f(t) = \sum_{s \le t} \mu(s,t)\, g(s)\end{array}}$$

__1st Pf__ For any t,

$$\sum_{s \le t} \mu(s,t) \left(\sum_{r \le s} g(r) \right) = \sum_r g(r) \sum_{r \le s \le t} \mu(s,t)$$

$$= \sum_r g(r) \left[\zeta \circ \mu \, (r,t) \right]$$

$$= \sum_r g(r)\, 1(r,t) = g(t) \quad \blacksquare$$

__2nd Pf__ $g = \zeta f \iff f = \mu g$

For details, see book $\blacksquare$

Over the next few classes we will discuss Möbius functions and inversion more slowly + combinatorially.

They are __very__ important