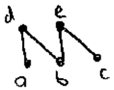
lecture 23
11.19.13

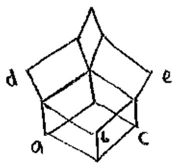
P is a lattice if every $x, y \in P$ have

- a least upper bound $x \vee y$.
- a greatest lower bound $x \wedge y$.

Ex: $[n]$, B_n, D_n, Π_n

- $J(P) = \{\text{order ideals of } P\}$ ordered by containment

$P =$ 

$J(P) =$ 

A lattice L is <u>distributive</u> if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x,y,z \in L$

$$\Updownarrow$$

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x,y,z \in L$

Theorem (Birkhoff)

L is distributive $\Leftrightarrow L \cong J(P)$ for some poset P

Pf

$\Leftarrow$: If I, J are order ideals, $I \cup J$, $I \cap J$ are order ideals.

Hence $I \vee J = I \cup J$, $I \wedge J = I \cap J$, so L is distributive.

$\Rightarrow$: Let L be distributive.

Let Q be the set of join-irreducibles.

$(x \neq \hat{0},\ x \neq y \vee z \text{ for } y \neq z)$

We claim $L \cong J(Q)$.

Define $\varphi : L \to J(Q)$

$t \mapsto I_t = \{s \in Q : s \leq t\}$

↑ an order ideal

Clear: $t \leq u \Rightarrow \varphi(t) \subseteq \varphi(u)$

φ injective

Note: $t = \bigvee_{\substack{s \in L \\ s \leq t}} s$

$$= \Big(\bigvee_{\substack{s \in Q \\ s \leq t}} s\Big) \vee \Big(\bigvee_{\substack{s \in L - Q \\ s \leq t}} s\Big)$$

↑ $s = s_1 \vee \cdots \vee s_k$ for $s_1, \ldots s_k$ join-irreds, $\leq t$

$$\boxed{t = \bigvee_{s \in I_t} s} \quad (*)$$

So $\varphi(t) = \varphi(u) \Rightarrow t = u$. ✓

Also: $\varphi(t) \subseteq \varphi(u)$
$\Downarrow$
$t \leq u$

φ surjective

Let $I \in J(Q)$.

Let $t = \bigvee_{i \in I} i$. Claim: $I = I_t$.

$\subseteq$: If $i \in I$ then $i \in Q, i \leq t \Rightarrow i \in I_t$.

$\supseteq$: Let $u \in I_t$. Since

$$t = \bigvee_{s \in I_t} s = \bigvee_{s \in I} s$$

$$t \wedge u = \bigvee_{s \in I_t} (s \wedge u) = \bigvee_{s \in I} (s \wedge u)$$

↑ One of the terms is u and the others are $\leq u$

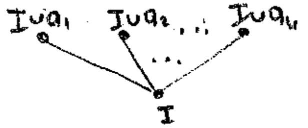
$$u = \bigvee_{s \in I} (s \wedge u)$$

Since u is join-irred, some $s \wedge u = u$ $(s \in I)$

so $u \leq s$ $(s \in I)$ so $u \in I$.

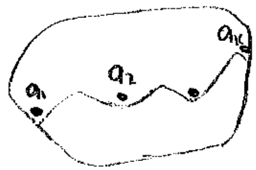
Method to draw J(P)

Key: "There are cubes everywhere" - certainly above and below each element.

If 

in J(P), there is a Boolean lattice $2^{\{a_1,\dots,a_k\}}$ above.

↕ ↕



$I\cup B$ is an order ideal for any $B\subseteq A$

So the method is to fill in these cubes sequentially.

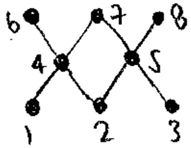
Dilworth's Theorem If the largest antichain of P has size w (the "width" of P) then $P = C_1\cup\dots\cup C_w$ for some chains $C_1,\dots,C_w$ of P

The Method:

① Decompose $P=C_1\uplus\dots\uplus C_w$ and assign a direction \ | / (of height 1) to each C_i.

② Choose a "linear extension" $\{a_1,\dots,a_n\}=P$ where $P_i=\{a_1,\dots,a_i\}$ is an order ideal for all i. ($a_i<a_j \Rightarrow i<j$)

③ For $i=1,2,\dots,n$, build $J(P_i)$ from P_{i-1} by:
- Look at element a_i, look for $P_{<a_i}$ in $J(P_{i-1})$, add an edge from it to new vertex $P_{\le a_i}$ in direction of appropr. chain
- Successively fill in all cubes above each vertex

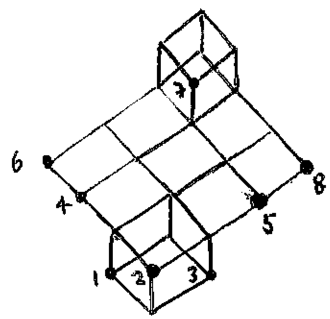
Ex: P = 

(labeling = linear extension)

$C_1=148$ $C_2=27$ $C_3=358$

We carry out the steps in colors: 12345678

↳ no meaning to the colors, just illustrating the step-by-step construction



HW: Do this for an interesting poset of ≥ 8 elements and width ≥ 3.

Thm J(P) is an induced subposet of $\mathbb{N}^w$ where w = width of P (Dilworth)

Pf: Exercise - based on construction above. ∎