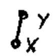
Posets

Lecture 22
11.14.13

A (partially ordered set)/(poset) is a set P with a binary relation $\leq$ such that

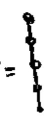
- refl. • $x \leq x$ for $x \in P$
- antisym. • $x \leq y$ and $y \leq x$ imply $x = y$ for $x, y \in P$
- trans. • $x \leq y$ and $y \leq z$ imply $x \leq z$ for $x, y, z \in P$

Say y covers x ($x \lessdot y$) if • $x < y$
• $\nexists z \in P$: $x < z < y$

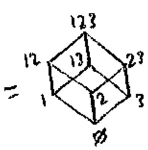
Draw  in "Hasse diagram"

Examples:

① For $n \in \mathbb{Z}$ let $\mathbf{n} = \{1, 2, \ldots, n\}$ with the usual order

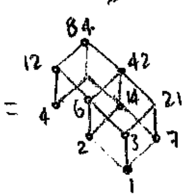
$\mathbf{5} =$ 

② Let B_n be the set of subsets of $[n]$ with $S \leq T$ if $S \subseteq T$

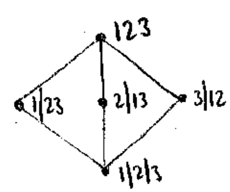
$B_3 =$ 

(skeleton of n-cube)

③ Let D_n be the set of divisors of n with $a \leq b$ if $a | b$

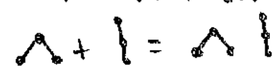
$D_{84} =$ 

④ Let Π_n be the set of partitions of $[n]$ with $\pi \leq \sigma$ if π refines σ: every block of π is a subset of a block of σ.

$\Pi_3 =$ 

Some defs:

- interval $[s,t] = \{u \in P : s \leq u \leq t\}$
- If there is a minimum elt, call it $\hat{0}$
  maximum $\hat{1}$
- P is locally finite if all its intervals are finite
- subposet:
  - induced: $Q \subseteq P$ where, for $x, y \in Q$, $x \leq_Q y \iff x \leq_P y$
  - weak: $Q \subseteq P$ $x \leq_Q y \Rightarrow x \leq_P y$
- $P \cong Q$ if there is a bijection $\varphi: P \to Q$ with $x \leq_P y \iff \varphi(x) \leq_Q \varphi(y)$.
- chain: $x_1 < x_2 < \cdots < x_k$
- antichain: $\{x_1, \ldots, x_k\}$ with x_i, x_j incomparable for all $i \neq j$.
- rank of P: size of longest chain
- P is ranked if for any $x \leq y$, all maximal chains from x to y have the same length. "It has levels"
- $P + Q$: poset on $P \cup Q$ with order rels from P, Q



- $P \times Q$: poset on $P \times Q$ with $(x,y) \leq_{P \times Q} (x', y') \iff$ $x \leq_P x'$ and $y \leq_Q y'$

- $I \subseteq P$ is a downset (order ideal) if $y \in I, x \leq y \Rightarrow x \in I$ | Similarly: upset (order filter)