Ex

Let r_n = # of Schröder paths from $(0,0)$ to $(2n,0)$ using steps ↗ → ↘ and never crossing below the x-axis.

(HW4.3: 1, 2, 6, 22, 90, 394, ...)

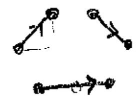
$$\begin{vmatrix} r_1 & r_2 & \cdots & r_n \\ r_2 & r_3 & \cdots & r_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_{n+1} & \cdots & r_{2n-1} \end{vmatrix} = 2^{\frac{n(n+1)}{2}}$$

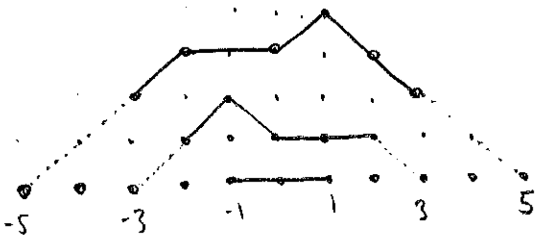
$|2| = 2$

$\begin{vmatrix} 2 & 6 \\ 6 & 22 \end{vmatrix} = 8$

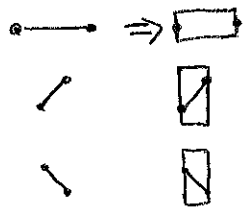
$\begin{vmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{vmatrix} = 64$

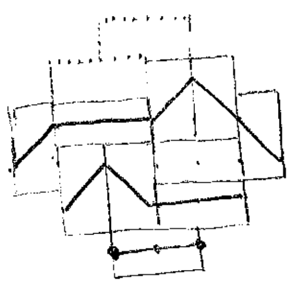
LHS = routings in this graph

edges: 

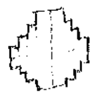


Now take each





A domino tiling of "Aztec diamond"!

 $= AD_4$

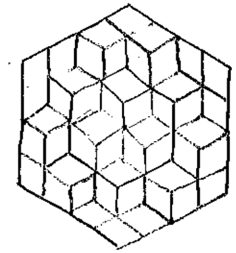
(Better: "Mayan diamond")

Thm (Elkies-Kuperberg-Larsen-Propp, Eu-Fu)
# of Schröder routings = # of dom. tilings of AD_n = $2^{\frac{n(n+1)}{2}}$

Lecture 21
11.12.13

More on tilings and determinants

Consider the TSSCPP of size n:



- totally symmetric
  (invariant under action of dihedral group)
- self-complementary
  (invariant under taking the complement in the box)
- plane partitions
  (stacks of cubes in the $n \times n \times n$ box)

Stanley: "A Baker's Dozen of Conjectures Concerning Plane Partitions" (1986)

Enum. of PPs by symmetry groups.

Conj: $\boxed{\#\, TSSCPP_n = \dfrac{1!\,4!\,7!\cdots(3n-2)!}{n!\,(n+1)!\cdots(2n-1)!}}$

Doran: path interpretation (late 80s)

Stembridge: determinantal formula (1990)

Andrews: evaluation of determinant (1994) ✓
(hypergeom fns, WZ)

An <u>alternating sign matrix</u> is an $n \times n$ matrix of 0s, 1s, -1s such that in any row and col, the 1s and -1s alternate, starting and ending with 1.

Ex:
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

<u>Conj</u> (Mills, Robbins, Rumsey 1983)

$$\# ASM_n = \frac{1!\,4!\,7!\cdots(3n-2)!}{n!\,(n+1)!\cdots(2n-1)!}$$

Proved by Zeilberger and Kuperberg in 1995.

No simple bijective proof for
- formula
- ASM $\longleftrightarrow$ TSSCPP

Kuperberg: Square ice model

```
H O H O H O H O H O H
 H   H   H   H   H
H O H O H O H O H O H
 H   H   H   H   H
H O H O H O H O H O H
 H   H   H   H   H
H O H O H O H O H O H
 H   H   H   H   H
H O H O H O H O H O H
```

$1 \mapsto$ HOH

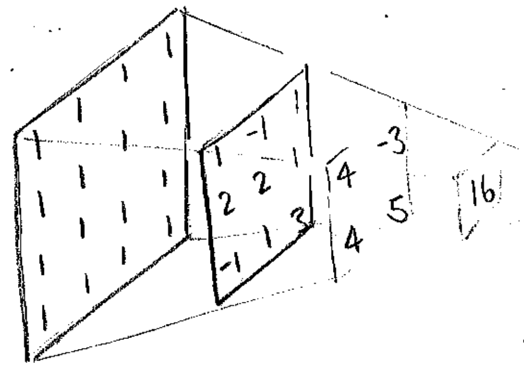
$-1 \mapsto$ H / O / H

$0 \mapsto$ HO / H

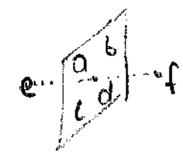
How were ASMs discovered?

<u>Dodgson condensation</u> (Lewis Carroll = Charles Dodgson)

A fast method to compute determinants.



$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} = 16$$



$$f = \frac{ad-bc}{e}$$

Combin. proof: Zeilberger '98

In variables

$$\begin{bmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{bmatrix} \to \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \to \begin{bmatrix} X_{11}X_{22}-X_{21}X_{12} & X_{12}X_{23}-X_{22}X_{13} \\ X_{21}X_{32}-X_{31}X_{22} & X_{22}X_{33}-X_{32}X_{23} \end{bmatrix}$$

$\downarrow$

$$\left[(X_{11}X_{22}-\underline{X_{21}X_{12}})(X_{22}X_{33}-\underline{X_{32}X_{23}})-(X_{21}X_{32}-X_{31}X_{22})(X_{12}X_{23}-X_{22}X_{13})\right]/X_{22}$$

This is the sum of 8 Laurent monomials, such as

$$\frac{X_{21}X_{12}X_{32}X_{23}}{X_{22}} \longleftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Corresponding to ASMs!

Each ASM appears $2^{\#(-1)s}$ times.

If we put y's in the first level, get

$$\begin{array}{|cccc|}Y_{11} & Y_{12} & Y_{13} & Y_{14}\\ Y_{21} & Y_{22} & Y_{23} & Y_{24}\\ Y_{31} & Y_{32} & Y_{33} & Y_{34}\\ Y_{41} & Y_{42} & Y_{43} & Y_{44}\end{array} \longrightarrow \begin{array}{|ccc|}X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\end{array} \longrightarrow \begin{array}{|cc|}\frac{X_{11}X_{22}-X_{21}X_{12}}{Y_{22}} & \frac{X_{12}X_{23}-X_{22}X_{13}}{Y_{23}}\\ \frac{X_{21}X_{32}-X_{31}X_{22}}{Y_{32}} & \frac{X_{22}X_{33}-X_{32}X_{23}}{Y_{33}}\end{array}$$

$$\downarrow$$

$$\left[\frac{(X_{11}X_{22}-X_{21}X_{12})}{Y_{22}}\frac{(X_{22}X_{33}-X_{32}X_{23})}{Y_{33}} - \frac{(X_{21}X_{32}-X_{31}X_{22})}{Y_{32}}\frac{(X_{21}X_{32}-X_{31}X_{22})}{Y_{23}}\right] \Big/ X_{22}$$

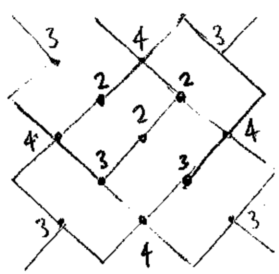
This is the sum of 8 monomials such as

$$\frac{X_{21}X_{12}X_{32}X_{23}}{X_{22}Y_{22}Y_{33}} \longleftrightarrow \begin{array}{ccccc}0 & & 1 & & 0\\ & -1 & & 0 & \\ 1 & & -1 & & 1\\ & 0 & & -1 & \\ 0 & & 1 & & 0\end{array}$$

corresponding to pairs of "compatible" ASMs of size $n, n-1$.

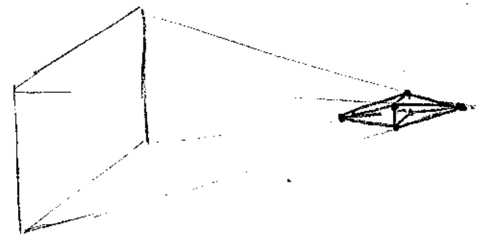
Now replace each number i with a vertex with its 3 edges NE, NW, SE, SW $\quad 1 \mapsto \quad -1 \mapsto \quad 0 \mapsto$

We get each tiling of the Mayan diamond MD_n exactly once!

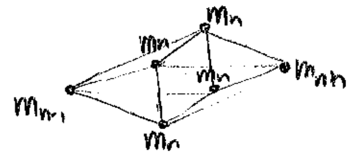


Corollary: MD_n has $2^{n(n+1)/2}$ domino tilings

Pf:



A term in the n-th row of Dodgson condensation has (m_n = # of tilings of MD_n) terms



In $ef = ad - bc$, the number of terms is

$$m_{n-1}m_{n+1} = m_n m_n + m_n m_n$$

So

$$\frac{m_{n+1}}{m_n} = 2\frac{m_n}{m_{n-1}} \Rightarrow \frac{m_n}{m_{n-1}} = 2^n \Rightarrow m_n = 2^{n+(n-1)+\dots+2+1}$$

Note: A priori it is <u>very surprising</u> that this rational recurrence produces Laurent monomials. This is part of the very interesting theory of cluster algebras.