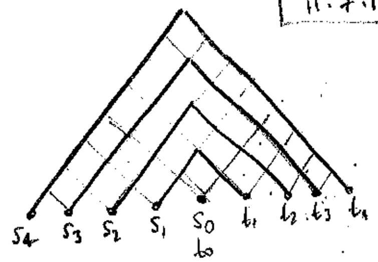
Lecture 20
11.7.13

Ex Let C_n = n-th Catalan number

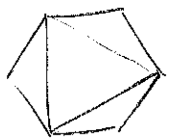
$$\begin{vmatrix} C_0 & C_1 & C_2 & \cdots & C_n \\ C_1 & C_2 & C_3 & \cdots & C_{n+1} \\ C_2 & C_3 & C_4 & \cdots & C_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n+1} & C_{n+2} & \cdots & C_{2n} \end{vmatrix} = 1$$



only one routing

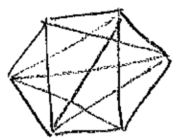
An interesting connection:

- A <u>triangulation</u> of a convex n-gon is a way of splitting it into triangles without introducing new vertices.



$$\begin{pmatrix} \text{\# of triangulations} \\ \text{of } (n+2)\text{-gon} \end{pmatrix} = C_n$$

- A <u>k-triangulation</u> is a maximal set of diagonals so that no k+1 of them cross pairwise. (k=1: triang)



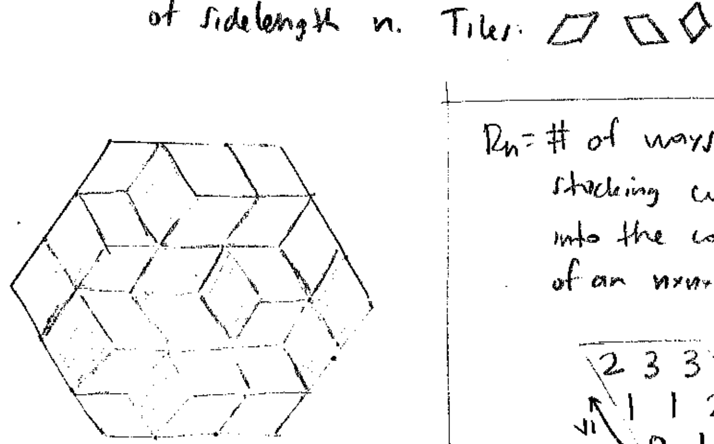
$$\begin{pmatrix} \text{\# of k-triangulations} \\ \text{of } (n+2)\text{-gon} \end{pmatrix} = \underbrace{\begin{vmatrix} C_n & C_{n-1} & \cdots & C_{n-k+1} \\ C_{n-1} & C_{n-2} & & C_{n-k} \\ \vdots & & & \vdots \\ C_{n-k+1} & C_{n-k} & & C_{n-2k+2} \end{vmatrix}}_{k}$$

= # of non-intersecting

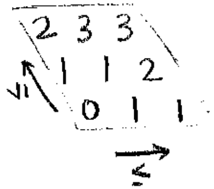
(Jonsson, Serrano-Stump)

Ex

Let R_n = # of rhombus tilings of a regular hexagon of sidelength n. Tiles:



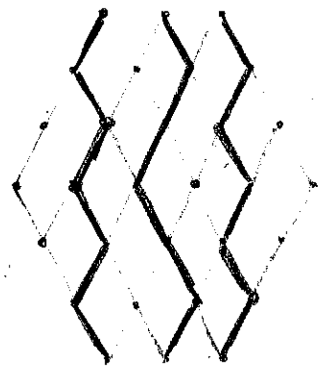
R_n = # of ways of stacking cubes into the corner of an $n \times n \times n$ box



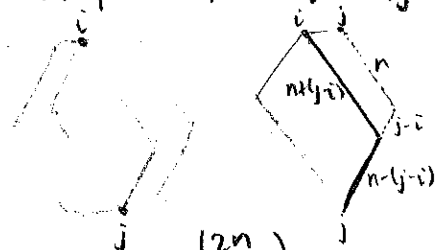
"plane partitions"

Cor: n^2 rhombi of each type

R_n = # of routings from top to bottom in G_n:



Note: paths from S_i to T_j



$\binom{2n}{n+i-j}$

$$R_n = \det \left| \binom{2n}{n+i-j} \right|_{0 \le i,j \le n-1} = \prod_{i=0}^{n} \frac{\binom{2n+i}{n}}{\binom{n+i}{n}} = \prod_{i,j,k=1}^{n} \frac{i+j+k-1}{i+j+k-2}$$

Ex: n=3

1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$\rightarrow R_3 = \begin{vmatrix} 20 & 15 & 6 \\ 15 & 20 & 15 \\ 6 & 15 & 20 \end{vmatrix} = 980$$

Ex

Let r_n = # of Schröder paths from $(0,0)$ to $(2n,0)$ using steps ↗ → ↘ and never crossing below the x-axis.

(HW4.3: $1, 2, 6, 22, 90, 394, \ldots$)

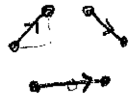
$$\begin{vmatrix} r_1 & r_2 & \cdots & r_n \\ r_2 & r_3 & \cdots & r_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_{n+1} & \cdots & r_{2n-1} \end{vmatrix} = 2^{\frac{n(n+1)}{2}}$$

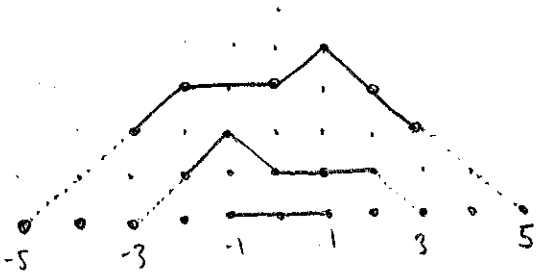
$|2| = 2$

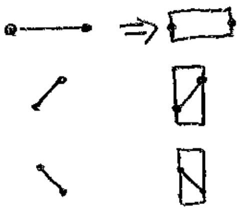
$\begin{vmatrix} 2 & 6 \\ 6 & 22 \end{vmatrix} = 8$

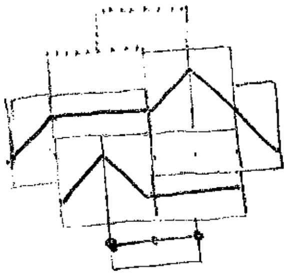
$\begin{vmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{vmatrix} = 64$

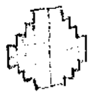
LHS = routings in this graph

edges: 



Now take each 



A domino tiling of "Aztec diamond"!  $= AD_4$

(Better: "Mayan diamond")

Thm (Elkies-Larson-Propp-Shor, Eu-Fu)

# of Schröder routings = # of dom. tilings of AD_n = $2^{\frac{n(n+1)}{2}}$