Lecture 2
9.3.13

Two basic counting principles

- Multiplication principle:

  If there are a ways of performing task A and b ways of performing task B (regardless of the outcome of A)

  then there are $a \cdot b$ ways of performing A, then B.

- Addition principle:

  If there are a ways of performing task A and b ways of performing task B

  then there are $a+b$ ways of performing one of A or B.

Permutations:

Let $S_n = \{$permutations of $\{1, .., n\}\}$

$\boxed{\text{Prop: } |S_n| = n! = n(n-1) \cdots 1}$

Pf To choose a permutation $\pi_1 \cdots \pi_n$:

- choose π_1 (n choices)
- choose π_2 ($n-1$ – all but π_1)

  $\vdots$ $\vdots$
- choose π_n (1 choice)

total: $n!$ choices ∎

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Subsets:

If S is a set of size n,

$2^S = \{$subsets of $S\}$

$\binom{S}{k} = \{k$-subsets of $S\}$ $0 \leq k \leq n$

$\boxed{\text{Prop } |2^S| = 2^n}$

Pf: Let $S = \{a_1 \dots a_n\}$. Choosing a subset $T \subseteq S$ is the same as choosing:

- Does $a_1 \in T$? (2 outcomes)
- Does $a_2 \in T$? (2 outcomes)

  $\vdots$ $\vdots$
- Does $a_n \in T$? (2 outcomes)

So I have $2 \cdot 2 \cdots 2 = 2^n$ outcomes ∎

Note this gives a bijection

$\{$subsets of $S\} \longleftrightarrow \{$seqs $(\varepsilon_1, .., \varepsilon_n) : \varepsilon_i = 0$ or $1\}$

Define $\binom{n}{k} := \left|\binom{S}{k}\right|$ for $|S| = n$

$=$ # of ways of choosing k elts from a set of n elts

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Prop $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Pf.

Let's count $\binom{S}{k}$ wrong:

To choose a subset $\{b_1, \dots b_k\}$ of $\{a_1, \dots a_n\}$.

- Choose b_1 (n choices)
- Choose b_2 ($n-1$ — all but b_1)

  ⋮

- Choose b_k ($n-(k-1)$ — all but $b_1, b_2, \dots b_{k-1}$)

Total choices: $n(n-1)\cdots(n-k+1)$.

But we wanted ordered k-sets. We showed

$$\begin{pmatrix}\#\text{ ordered } k\text{-subsets} \\ \text{of } \{a_1 \dots a_n\}\end{pmatrix} = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!} \qquad (*)$$

Now count them differently. To choose an ordered k-subset of $\{a_1 \dots a_n\}$:

- choose an unordered k-subset ($\binom{n}{k}$ choices)
- order it ($k!$ choices)

So

$$\begin{pmatrix}\#\text{ ordered } k\text{-subsets} \\ \text{of } \{a_1 \dots a_n\}\end{pmatrix} = \binom{n}{k} k! \qquad (**)$$

We counted the same quantity in two (correct) ways, so combining $(*)$, $(**)$ we get

$$\frac{n!}{(n-k)!} = \binom{n}{k} k!$$

∎

Subsets and GFs:

The multivariate GF for subsets of $[n] = \{1, \dots n\}$ is

$$\sum_{A \subseteq [n]} \prod_{a \in A} x_a = (1+x_1)(1+x_2)\cdots(1+x_n)$$

by inspection or by induction.

$n=2$: $1 + x_1 + x_2 + x_1 x_2$

Now let $x_1 = \dots = x_n = x$:

$$\sum_{A \subseteq [n]} x^{|A|} = (1+x)^n$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n}$$ Binomial theorem

Note: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ from combin, alg.

Exercise: $\sum_{k=0}^{n} k \binom{n}{k} = n 2^{n-1}$ from combin, alg.

Compositions:

A composition of n is a way of expressing n as an ordered sum of positive integers

$n=3$: $1+1+1$ $\quad 2+1$ $\quad 1+2$ $\quad 3$

A k-comp is one with k parts

A weak comp is one into non-negative integers.

Prop There are 2^{n-1} compositions of n

$\binom{n-1}{k-1}$ k-comps of n

Pf To choose a comp of n

- write $n = 1+1+1+1+\cdots+1+1$
- delete some of the +s (2^{n-1} choices)
- group consecutive 1s into one part

(Ex: $8 = 1+1\cancel{+}1\cancel{+}1+1+1\cancel{+}1+1 \Rightarrow 8 = 1+3+2+2$)

Similarly for k-comps. ∎

Prop There are $\binom{n+k-1}{k-1}$ weak k-comps of n

Pf $n = a_1 + \cdots + a_k$ is a weak k-comp of n

$$\Updownarrow$$

$n+k = (a_1+1)+\cdots+(a_k+1)$ is a k-comp of $n+k$. ∎

Multisets

A <u>multiset</u> is a set with possibly repeated elements, like $\{1,2,2,2,4,5,5,7\} = \{1^1, 2^3, 4^1, 5^2, 7^1\}$ ← notation

This multiset has cardinality/size 8.

Let $\left(\!\binom{S}{k}\!\right) = \{k\text{-multisets on } S\}$. Let $\left(\!\binom{n}{k}\!\right) = \left|\left(\!\binom{S}{k}\!\right)\right|$ for $|S| = n$.

↑ elts. chosen from S

Prop $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

Pf If a k-multiset S on $[n]$ has a_i copies of i $(1 \le i \le n)$ then $a_1 + \cdots + a_n = k$ is a weak n-composition of k, and viceversa. So

$$\left(\!\binom{n}{k}\!\right) = \binom{k+n-1}{n-1} = \binom{n+k-1}{k} \quad ∎$$

Multisets and GFs

The multivariate GF for multisets on $[n]$ is

$$\sum_{v:[n]\to\mathbb{N}} \prod_{i=1}^{n} x_i^{v(i)} = (1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots(1+x_n+x_n^2+\cdots)$$

Again letting $x_1 = \cdots = x_n = x$,

$$\sum_{v:[n]\to\mathbb{N}} x^{v(1)+\cdots+v(n)} = (1+x+x^2+\cdots)^n$$

$$\sum_{\substack{M \text{ multiset} \\ \text{on } [n]}} x^{|M|} = \left(\frac{1}{1-x}\right)^n$$

binomial thm

$$\sum_{k=0}^{\infty} \left(\!\binom{n}{k}\!\right) x^k = (1-x)^{-n} =: \sum_{k\ge 0} \binom{-n}{k}(-x)^k$$

So

$$\boxed{\left(\!\binom{n}{k}\!\right) = (-1)^k \binom{-n}{k}}$$ "combin. reciprocity thm"

The <u>multinomial coefficient</u> $\binom{n}{a_1,\dots,a_m}$ is the number of ways of splitting an n-set into an a_1-set, an a_2-set, .., and an a_m-set in order (where $a_1+\dots+a_m=n$)

For ex. $\binom{n}{k,\,n-k}=\binom{n}{k}$

Prop The number of permutations of $\{1^{a_1},2^{a_2},\dots,m^{a_m}\}$ is $\binom{n}{a_1,\dots,a_m}$ where $a_1+\dots+a_m=n$

Pf Out of the $a_1+\dots+a_m$ positions - - - - - - -

choose which a_1 of them will hold the 1s

a_2 2s

$\vdots$

a_m ms ∎

Prop $\binom{n}{a_1,\dots,a_m}=\dfrac{n!}{a_1!\cdots a_m!}$

Prop $(x_1+\dots+x_m)^n=\sum\limits_{\substack{a_1+\dots+a_m=n\\ a_i\geq 0}}\binom{n}{a_1\dots a_m}x_1^{a_1}\cdots x_m^{a_m}$

Prop In the m-dim. box of dimensions $a_1\times\dots\times a_m$, there are $\binom{n}{a_1\dots a_m}$ shortest lattice paths from $(0,..,0)$ to $(a_1,..a_m)$

$\binom{7}{3,2,2}$