Another important source of such formulas:

Let

$$0 \longrightarrow V_n \xrightarrow{\partial_n} V_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} W \longrightarrow 0$$

be an exact sequence of fin-dim vector spaces; that is, $\operatorname{im} \partial_{j+1} = \ker \partial_j$. Then

$$\boxed{\dim W = \sum_{i=0}^{n} (-1)^i \dim V_i}$$

Pf Induct on n.

Lecture 19
11.05.13

$$\beta_n(S_1 \ldots S_k) = \begin{vmatrix} \binom{S_1}{0} & \binom{S_2}{0} & \binom{S_3}{0} & \cdots & \binom{n}{0} \\ \binom{S_1}{S_1} & \binom{S_2}{S_1} & \binom{S_3}{S_1} & \cdots & \binom{n}{S_1} \\ 0 & \binom{S_2}{S_2} & \binom{S_3}{S_2} & \cdots & \binom{n}{S_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{n} \end{vmatrix}$$

More generally, consider the "binomial determinant"

$$\det\left(\binom{a_i}{b_j}\right)_{1\le i,j\le n}$$

for $0 \le a_1 < \cdots < a_n$ integers
$0 \le b_1 < \cdots < b_n$

These determinants appeared in algebraic geometry (Lascoux - Classes de Chern d'un produit tensoriel)

Gessel-Viennot: why are they positive?

Lindström - Gessel - Viennot Lemma (Karlin - McGregor)

Let G be a directed graph with no directed cycles.

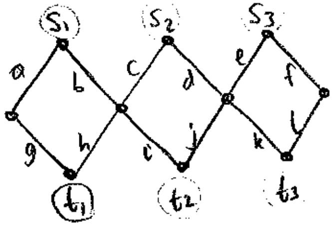
Let $s_1 \ldots s_n$ be "sources"
$t_1 \ldots t_n$ be "sinks"

Assume all routings (n vertex-disjoint paths) from $\{s_1 \ldots s_n\} = S$ to $\{t_1 \ldots t_n\} = T$ connect s_1 to $t_1, \ldots, s_n$ to t_n. Then

$$\det(a_{ij})_{1\le i,j\le n} = \#\text{. of routings from } S \text{ to } T.$$

- There is a version allowing cycles
- There is a version with edge weights: $a_{ij} = \sum_{P \text{ path } i \to j} \operatorname{wt}(P)$, $\operatorname{wt}(P) = \prod_{e\in P} \operatorname{wt}(e)$
  $\det(a_{ij}) = \sum_{R \text{ routing}} \operatorname{wt}(R)$
- If other pairings are possible, $\det(a_{ij}) = \sum_{R \text{ routings}} \operatorname{sgn}(R) \operatorname{wt}(R)$.

Ex.



↓

$$\det\begin{vmatrix} ag+bh & bi & 0 \\ ch & ci+dj & dk \\ 0 & ej & ek+fl \end{vmatrix} =$$

$= agciek + agcifl + agdjek + agdjfl + bhciek + bhcifl + bhdjek + bhdjfl$

$- agdkej - bhdkej - bichek - bichfl$ = ⟨⟨⟨ + ⟨⟨⟩ + ⟨⟩⟩ + ⟩⟩⟩

⟨X ⟩X X⟨ X⟩

Pf

$$\det(a_{ij}) = \sum_{\pi\in S_n} \operatorname{sgn}\pi \; a_{1\pi(1)}\cdots a_{n\pi(n)}$$

$$= \sum_{\pi\in S_n} \operatorname{sgn}\pi \Big(\sum_{P_1:1\to\pi(1)} \operatorname{wt}(P_1)\Big)\cdots\Big(\sum_{P_n:n\to\pi(n)} \operatorname{wt}(P_n)\Big)$$

$$= \sum_{P:S\to T} \operatorname{sgn} P \operatorname{wt} P$$

over all "path systems" P from S to T.

- Every routing arises exactly once: 
- Every nonrouting  is cancelled by  where we flip the first intersection of the first P_i (min i) intersecting some P_j $(j>i)$.

Cor

Let $0\le a_1<\cdots<a_n$ be integers

$0\le b_1<\cdots<b_n$

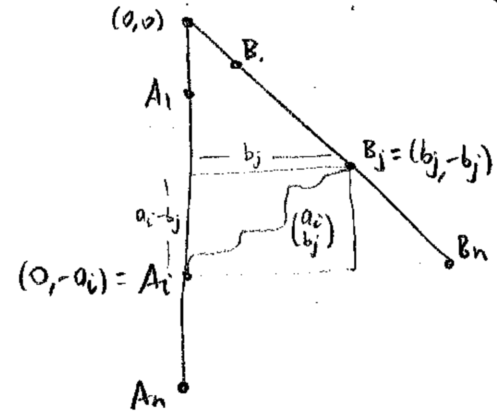
Let $A_i := (0,-a_i)$, $B_i := (b_i,-b_i)$

Orient edges ↑, →.

Then there are

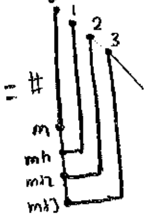
$$\det\left(\binom{a_i}{b_j}\right)_{1\le i,j\le n}$$

routings from $\{A_1\ldots A_n\}$ to $\{B_1\ldots B_n\}$



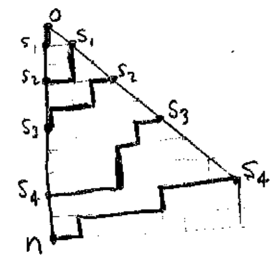
Cor: This det ≥ 0

Ex
$$\begin{vmatrix} \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{n} \\ \binom{m+1}{0} & \binom{m+1}{1} & \cdots & \binom{m+1}{n} \\ \vdots & & & \\ \binom{m+n}{0} & \binom{m+n}{1} & & \binom{m+n}{n} \end{vmatrix} = \#$$



$= 1$

Ex
$$\begin{vmatrix} \binom{s_1}{0} & \binom{s_2}{0} & \cdots & \binom{n}{0} \\ \binom{s_1}{s_1} & \binom{s_2}{s_1} & \cdots & \binom{n}{s_1} \\ \vdots & & & \\ \binom{s_1}{s_k} & \binom{s_2}{s_k} & & \binom{n}{s_k} \end{vmatrix} = \#$$



= # perms of $[n]$ with descents at $s_1,\ldots,s_k$

↑

bijection?