Sieve Methods

Lecture 18
10·31·13

EC1, Ch. 2

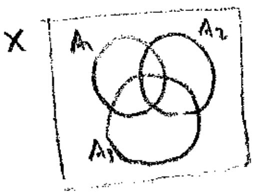
A sieve method is a method for enumerating objects by overcounting, and then subtracting off unwanted elements.

Eratosthenes's sieve:

To list primes from 2 to n:

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Based on simple idea:



$$|X \setminus (A_1 \cup A_2 \cup A_3)| = |X| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

$$\boxed{\left|X \setminus \bigcup_{i=1}^{n} A_i\right| = |X| - \sum_{1 \le s_1 < \cdots < s_k \le n} (-1)^k |A_{s_1} \cap \cdots \cap A_{s_k}|}$$

Pf: If x is in t sets, it is counted

$$1 - t + \binom{t}{2} - \binom{t}{3} + \cdots = (1-1)^t \text{ times} \qquad \blacksquare$$

Principle of Inclusion-Exclusion

Let A be a finite set

Let S be a finite set of properties that elts. of A may or may not have. For $T \subseteq S$,

Let $f_=(T)$ = # elts of A satisfying exactly the properties in T

$f_\geq(T)$ # elts of A satisfying the properties of T (and maybe more)

$$= \sum_{U \supseteq T} f_=(U)$$

Then $\boxed{f_=(T) = \sum_{U \supseteq T} (-1)^{|U-T|} f_\geq(U)}$

In particular, the # of elts of A satisfying no properties in S is

$$f_=(\emptyset) = \sum_{U \subseteq S} (-1)^{|U|} f_\geq(U)$$

The point: It is often easier to count objects <u>with</u> certain given properties.

Dual formulation:

$$f_=(T) = \sum_{U \subseteq T} (-1)^{|T-U|} f_\leq(U)$$

Ex 1 Derangements

Let D_n = # of perms of $[n]$ such that $\pi(i)\neq i$ for all i.

Let $S=\{S_1,\dots,S_n\}$ where S_i = property that $\pi(i)\neq i$

Then

$$D_n = f_=(\emptyset) = \sum_{T\subseteq[n]} (-1)^{n-|T|} f_\geq(T)$$

$$= \sum_{T\subseteq[n]} (-1)^{n-|T|}\,(n-|T|)!$$

$$= \sum_{t=0}^{n} \binom{n}{t} (-1)^{n-t}\,(n-t)!$$

$$= \sum_{t=0}^{n} (-1)^{n-t}\frac{n!}{t!} \sim \frac{n!}{e}$$

Ex 2 Euler's totient function

Let $\varphi(n)$ = # of numbers $1\le k\le n$ with $\gcd(n,k)=1$.

Let $n=p_1^{\alpha_1}\cdots p_m^{\alpha_m}$ Then

$$\gcd(n,k)=1 \iff p_i \nmid n \quad \text{for } i=1,\dots,m$$

Then

$$\varphi(n) = \sum_{i_1\dots i_t\subseteq[m]} (-1)^{m-t} \left(\begin{array}{c}\text{\# of numbers } 1\le k\le n \text{ such that}\\ p_{i_1}|n,\dots,p_{i_t}|n\end{array}\right)$$

$$= \sum_{i_1\dots i_t\subseteq[m]} (-1)^{m-t}\frac{n}{p_{i_1}\cdots p_{i_t}}$$

$$= n\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_t}\right)$$

Ex 3 Permutations by descents

Recall that the descent set of a permutation π is $D(\pi)=\{i:\pi(i)>\pi(i+1)\}$, and $\operatorname{des}(\pi)=|D(\pi)|$.

We counted perms by # of descents (Eulerian polynomial)

Now we count them by set of descents.

Let $\alpha_n(S)$ = # perms of $[n]$ with $\operatorname{Des}(\pi)\subseteq S$

$\beta_n(S)$ = # perms of $[n]$ with $\operatorname{Des}(\pi) = S$

Then

$$\alpha_n(S)=\sum_{T\subseteq S}\beta_n(T) \qquad \beta_n(S)=\sum_{T\subseteq S}(-1)^{|S-T|}\alpha_n(T)$$

Now:

$$\alpha_n(\{s_1<s_2<\dots<s_k\}) = \binom{n}{s_1,\ s_2-s_1,\ s_3-s_2,\ \dots,\ n-s_k}$$

Choose the $s_{i+1}-s_i$ numbers in positions $s_i+1,\dots,s_{i+1}$, put them in increasing order.

So

$$\beta_n(s_1\dots s_k) = \sum_{1\le i_1<\dots<i_j\le k} (-1)^{k-j}\binom{n}{s_{i_1},\ s_{i_2}-s_{i_1},\ \dots,\ n-s_{i_j}}$$

(letting $s_{k+1}=n$)

$$= n!\begin{vmatrix} \frac{1}{s_1!} & \frac{1}{s_2!} & \frac{1}{s_3!} & \cdots & \frac{1}{s_{k+1}!} \\ 1 & \frac{1}{(s_2-s_1)!} & \frac{1}{(s_3-s_1)!} & \cdots & \frac{1}{(s_{k+1}-s_1)!} \\ 0 & 1 & \frac{1}{(s_3-s_2)!} & \cdots & \frac{1}{(s_{k+1}-s_2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{(s_{k+1}-s_k)!} \end{vmatrix}$$