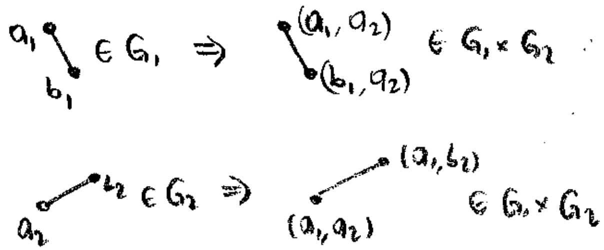
lecture 17
10.29.13

Note: If $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$ then $G_1\times G_2$ is the graph on $V_1\times V_2$ with edges



Easy: $C_n = (\bullet\!-\!\bullet)^n$

Prop. If $L(G_1)$ has eigenvalues $\lambda_1,\dots,\lambda_a$
$L(G_2)$ has eigenvalues $\mu_1,\dots,\mu_b$
then $L(G_1\times G_2)$ has eigenvalues $\lambda_i+\mu_j$ $\quad 1\le i\le a$, $1\le j\le b$

Pf Take eigenvectors r of $L(G_1)$ with eigenval λ
s of $L(G_2)$ with eigenval μ

Let $t=(r_i s_I)_{\substack{1\le i\le a\\ 1\le I\le b}}$. Then

$$1\le i\le a \Rightarrow (\deg v_i) r_i - \sum_{i - j} r_j = \lambda r_i$$

$$1\le I\le b \Rightarrow (\deg w_I) s_I - \sum_{I - J} s_J = \mu s_I$$

The entry (i,I) of $L(G_1\times G_2)t$ is

$$\underbrace{(\deg v_i+\deg w_I)}_{=\deg(v_i,w_I)} r_i s_I - \sum_{i - j} r_j s_I - \sum_{I - J} r_i s_J = r_i(\mu s_I) + s_I(\lambda r_i) = (\lambda+\mu) r_i s_I.$$

Cor The eigenvalues of C_n are

$$0, \underbrace{2,\dots,2}_{\binom{n}{1}\text{ times}}, \underbrace{4,\dots,4}_{\binom{n}{2}\text{ times}}, \dots, \underbrace{2n-2,\dots,2n-2}_{\binom{n}{n-1}\text{ times}}, 2n$$

Pf Induction

Thm The number of spanning trees of C_n is

$$\boxed{2^{2^n-n-1}\cdot 1^{\binom{n}{1}} 2^{\binom{n}{2}} \cdots (n-1)^{\binom{n}{n-1}} n^{\binom{n}{n}}}$$

Pf The matrix tree theorem gives

$$\frac{1}{2^n}\cdot 2^{\binom{n}{1}} 4^{\binom{n}{2}} \cdots (2n)^{\binom{n}{n}} \qquad \blacksquare$$

No combinatorial proof known! (See project list for a generalization)

Pf of 2nd formulation of matrix-tree-theorem:

The characteristic polynomial of the Laplacian is:

$$\det(L-\lambda I) = \begin{vmatrix} d_1-\lambda & & \\ & d_2-\lambda & \\ & & \ddots \\ & & & d_n-\lambda \end{vmatrix} = (\lambda_1-\lambda)\cdots(\lambda_{n-1}-\lambda)(0-\lambda)$$

The coeff of $-\lambda$ is the sum of the dets of the n ppal cofactors, which are equal • $\lambda_1\cdots\lambda_m$

<u>Matrix-tree theorem</u> (directed version)

Let D be a directed graph on $[n]$. ("digraph")

The Laplacian of D is

$$L_{ij} = \begin{cases} -(\#\text{ edges } i \to j) & i \neq j \\ \text{outdegree}(i) - (\#\text{ loops at } i) & i = j \end{cases}$$

A <u>directed spanning tree rooted at v</u> is one where all edges point toward v.

$$\begin{pmatrix} \#\text{ directed spanning trees} \\ \text{rooted at } v \end{pmatrix} = \det(v\text{-th ppal cofactor}) = \frac{1}{n}\lambda_1 \cdots \lambda_{n-1}$$

Corollary: This is independent of v!

<u>Eulerian walks</u>

An <u>Eulerian walk</u> in a digraph D is a closed walk which uses every edge exactly once. If such a walk exists, D is Eulerian.

<u>Prop</u> A connected digraph is Eulerian $\iff$ indeg(v) = outdeg(v) for all v

<u>Pf.</u> "Just walk"

- Start walking from v_0. Since indeg(v) = outdeg(v), every time I enter $v \neq v_0$ I can exit it. So I can only get stuck at v_0. Call this walk D_1.
- If D_1 is not Eulerian, there is a missing edge $u \to v$ with u in D_1. Walk until you get stuck (at u), call this new walk D_2.

Note that we can merge $D_1 \cup D_2$ into a single walk.
- If $D_1 \cup D_2$ is not Eulerian, repeat.

Eventually we will get an Eulerian walk. ∎

With a bit of advanced planning to make sure we cover the whole graph, we get a stronger result

Theorem Let D be an Eulerian graph on $[n]$. Consider an edge $\underset{v}{\bullet}\!\xrightarrow[e]{}$ Let

$\tau(D,v) = \#$ oriented spanning trees of D rooted at v. (indep of v)

$\epsilon(D,e) = \#$ Eulerian walks of D starting at e (indep of e)

Then

$$\epsilon(D,e) = \tau(D,v)\prod_{i=1}^{n}(\text{outdeg}(i)-1)!$$

Pf Let T be one such tree.

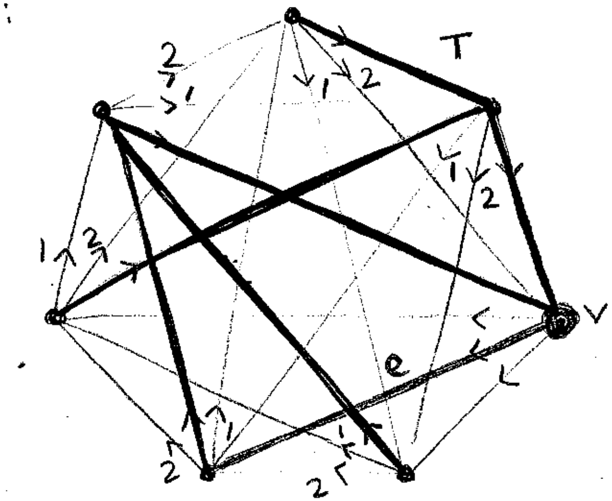
Each vertex i has a unique edge on T pointing towards v. Linearly order the other $\text{outdeg}(i)-1$ arbitrarily.

(For $i=v$, order the outedges $\neq e$).

We claim this gives an Eulerian walk E by:

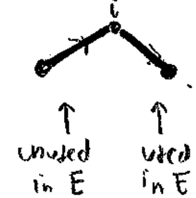
- Start with e
- Every time you enter a vertex i, leave using the lowest unused out-edge. (If none available, use the out-edge in T.)

Ex:



Why is E Eulerian?

E can only get stuck at v. If there are unused edges, then there are unused edges in T, and then there is



But then we entered v $<\text{indeg}(v)$ times and exited it $\text{outdeg}(i)$ times.

Conversely, given an Eulerian walk E, it is not hard to see that $\{\text{last edge exited from } i : i \neq v\}$ is an oriented spanning tree rooted at v. ∎

Ex. $D = \overset{\leftrightarrows}{K_n} =$ 

$\#$ of Eulerian walks $= n^{n-2}\cdot[(n-2)!]^n$