Counting spanning trees

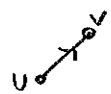
Lecture 16
10·24·13

A **spanning tree** T of a conn. graph G is a set of edges which connects all vertices without creating any cycles.

Easy: If G has n vertices, T has $n-1$ edges.

Given a graph $G=(V,E)$, orient edges arbitrarily.

- The incidence matrix M is the $V \times E$ matrix with



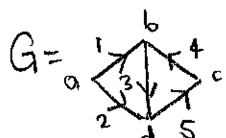
$$\begin{array}{c|c} & e \\ \hline u & 1 \\ v & -1 \end{array}$$

- The Laplacian matrix $L = MM^T$ is the $V \times V$ matrix where

$$L_{uv} = \begin{cases} -(\text{\# edges conn. } u \text{ and } v) & u \neq v \\ \deg v & u = v \end{cases}$$

(indep. of orientation)

Ex: $G =$



$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & & & \\ -1 & & 1 & -1 & \\ & & & 1 & -1 \\ & -1 & -1 & & 1 \end{bmatrix} \end{array}$$

$$L = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \end{array}$$

Note: L is symmetric
L is singular

A **principal cofactor** of L is a matrix obtained by removing row i, col i.

Matrix-Tree Theorem (Kirchhoff).
The number of spanning trees of a conn. graph G equals the determinant of any ppal cofactor of the Laplacian $L(G)$.
If the eigenvalues of $L(G)$ are $\lambda_1, \dots, \lambda_{n-1}, \lambda_n = 0$, this equals $\frac{1}{n}\lambda_1 \lambda_2 \cdots \lambda_{n-1}$.

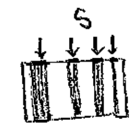
Pf In three steps.

Step 1. Binet-Cauchy Thm

Let $A = {}_m\overset{n}{\boxed{}}$, $B = {}_n\overset{m}{\boxed{}}$ with $m<n$. Then

$$\det(AB) = \sum_{\substack{S \subset [n] \\ |S|=m}} \det A[S] \det B[S]$$

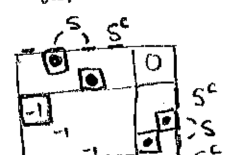
$A(S) =$



Pf.

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ -I & B \end{bmatrix} = \begin{bmatrix} 0 & AB \\ -I & B \end{bmatrix}$$

$\det = 1$; $\det = \pm$ RHS ; $\det = \pm AB$



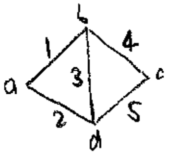
Step 2 Let G be a graph

Let M_0 be its adj. mx. with last row removed.

Then the max minors are

$$\det M_0[S] = \begin{cases} \pm 1 & \text{if } S \text{ is a spanning tree} \\ 0 & \text{otherwise} \end{cases}$$

Pf.



$$M_0 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline a & 1 & 1 & & & \\ b & -1 & & 1 & -1 & \\ c & & & & 1 & -1 \\ d & & -1 & -1 & & 1 \end{array}$$

If $V-1$ edges don't give a spanning tree, they form a cycle (like 123). In M_0:
$$\begin{array}{ccc} 1 & 2 & 3 \\ \hline 1 & 1 & \\ -1 & & 1 \\ & -1 & -1 \end{array}$$
rows add to 0, so $\det = 0$

If they do (like 134) then compute inductively

$$\det \begin{array}{c|ccc} & 1 & 3 & 4 \\ \hline a & 1 & & \\ b & -1 & 1 & -1 \\ c & & & 1 \end{array} \underset{\substack{\uparrow \\ \text{incident} \\ \text{to } d}}{=} \pm \det \begin{array}{c|cc} & 1 & 4 \\ \hline a & 1 & -1 \\ c & & 1 \end{array} \underset{\substack{\uparrow \\ \text{incident} \\ \text{to } b}}{=} \pm \det \begin{array}{c|c} & 4 \\ \hline a & -1 \end{array} = \pm 1$$

where each step is an expansion by cofactors of a col. with a single ± 1. ■

Step 3.

$$\det\begin{pmatrix}\text{Ppal cofactor} \\ \text{of } L_0\end{pmatrix} = \det(M_0 M_0^T) = \sum_{\substack{S \\ |S|=V-1}} \det M_0[S] \det M_0^T[S]$$

$$= \sum_{|S|=|V|-1} \det M_0[S]^2$$

$$= \sum_{\substack{\text{spanning} \\ \text{trees}}} 1 \quad ■$$

Ex 1 $K_n =$  complete graph

$$L(K_n) = \underbrace{\begin{bmatrix} n-1 & -1 & & -1 \\ -1 & n-1 & & -1 \\ & & \ddots & -1 \\ -1 & -1 & & n-1 \end{bmatrix}}_{n \times n} = nI - J \qquad I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Pf 1.

$$\det L_0 = \underbrace{\begin{vmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & & \\ -1 & -1 & \cdots & n-1 \end{vmatrix}}_{(n-1)\times(n-1)} \underset{\substack{\uparrow \\ \text{add all} \\ \text{rows to row 1}}}{=} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & & & \vdots \\ -1 & -1 & \cdots & n-1 \end{vmatrix} \underset{\substack{\uparrow \\ \text{add row 1} \\ \text{to all rows}}}{=} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & \cdots & 0 \\ & & & \vdots \\ 0 & 0 & \cdots & n \end{vmatrix} = n^{n-2}$$

Pf 2. J has eigenvectors $e_1-e_2, e_1-e_3, \dots, e_{n-1}-e_n$ of eigenvalue 0

Since $\operatorname{tr} J = n$, the other eigenvalue of J is n.

If $J \cdot v = \lambda v \Rightarrow (nI - J)v = nv - \lambda v = (n-\lambda)v$

So eigenvalues of L are $n, n, \dots, n, 0$

$$\Rightarrow \frac{1}{n}\lambda_1 \cdots \lambda_{n-1} = \frac{1}{n} \cdot n^{n-1} = n^{n-2}.$$

Ex 2 (Exercise) $K_{m,n} =$ 

\# spanning trees $= m^{n-1} n^{m-1}$

(combin. proof?)

Ex 3. The <u>n-cube</u> C_n has 2^n vertices $(a_1, \dots, a_n)$ $a_i = 0$ or 1.

Two vertices a, b are adjacent if they differ in one coord.

$C_0 =$  $C_1 =$  $C_2 =$  $C_3 =$  …