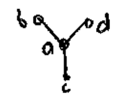
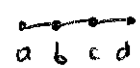
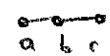
Lecture 15
10·17·13

Trees

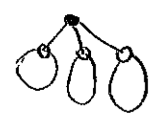
A <u>tree</u> on $[n]$ is a graph on vertices $1 \ldots n$ which is connected and has no cycles. $t(n) = \#$ of them

$n=4$:   $n=3$: 

4 + 12 = 16 | 3

A <u>rooted tree</u> is a tree with a chosen vertex called the root. $r(n) = \#$ of them RT = comb. class

$$r(n) = n\,t(n).$$

Note: A rooted tree is  a root and a "rooted forest"

- RT = Atom ★ RF
- RF = Set (RT)

So

$$R(z) = z\,e^{R(z)}$$

One interpretation: $R(z) = ze^{ze^{ze^{ze^{\cdots}}}}$

Another: $z = R(z)e^{-R(z)}$

$$\Downarrow$$

$R(z) = (ze^{-z})^{\langle -1 \rangle}$ (compositional inverse).

How do I find the coeffs. of a comp inverse?

<u>Fact</u>

A power series $f(x) = a_1x + a_2x^2 + \dots \in \mathbb{R}[[x]]$ has a compositional inverse $f^{-1}(x)$ iff $a_1 \neq 0$, in which case $f^{\langle -1 \rangle}(f(x)) = f(f^{\langle -1 \rangle}(x)) = x$

<u>Lagrange Inversion Formula</u>: If $f^{\langle -1 \rangle}(x)$ exists,

$$\boxed{n[x^n]\left(f^{\langle -1 \rangle}(x)\right)^k = k[x^{n-k}]\left(\frac{x}{f(x)}\right)^n}$$

Then

$$\begin{aligned} n\frac{r(n)}{n!} &= n[x^n]R(x) \\ &= n[x^n](xe^{-x})^{\langle -1 \rangle} \\ &= [x^{n-1}]\left(\frac{x}{xe^{-x}}\right)^n \\ &= [x^{n-1}]e^{nx} \\ &= \frac{n^{n-1}}{(n-1)!} \end{aligned}$$

Sylvester-Cayley

$$\Rightarrow \boxed{\begin{aligned} r(n) &= n^{n-1} \\ t(n) &= n^{n-2} \end{aligned}}$$

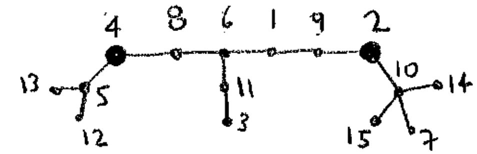
How to prove Lagrange inversion?

- analysis
- combinatorially, using combin. of trees!

Such a nice formula deserves a bijective proof!

① Look up "Prüfer code"

② Need to show there are n^n "trees with skeleton"

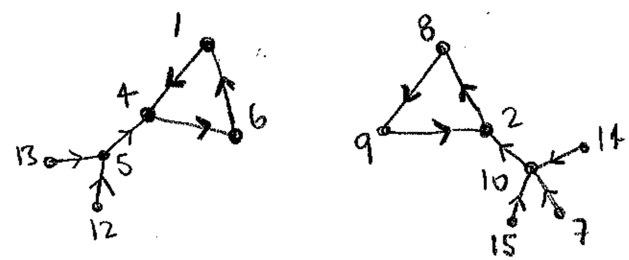


$$\begin{pmatrix} 1 & 2 & 4 & 6 & 8 & 9 \\ 4 & 8 & 6 & 1 & 9 & 2 \end{pmatrix}$$

$$=$$

$$(146)(289)$$

$$\downarrow$$



Turn the permutation of the skeleton into cycle notation, draw the graph of the permutation, "rehang" the trees hanging from the skeleton, and point all arrows towards the cycles.

- The result is the graph of a function $f: [n] \to [n]$ and this mapping is a bijection.

Parking functions

There are n cars $C_1 \dots C_n$ trying to park on the n spots $1 \dots n$ of a one way street.

Car C_i has a preferred spot a_i; it takes that one, or the first available after it.

Ex: $a=(3,2,3,1,2) \Rightarrow$ 

$a=(4,3,3,4,1) \Rightarrow$ 

… ?

a is a <u>parking function</u> if all cars can park.

Prop a is a parking function $\Updownarrow$ it contains $\geq i$ numbers $\leq i$ for all i.

Equiv: Reordering it to get $b_1 \leq b_2 \leq \dots \leq b_n$, I need $b_i \leq i$ for all i.

$\Downarrow$ If not, there are $> n-i$ numbers $> i$ $\Rightarrow$ $> n-i$ cars fighting for $n-i$ spots

$\Uparrow$ Exercise

Ex: $n=3$:

111
112 121 211
113 131 311
122 212 221
123 132 213 231 312 321

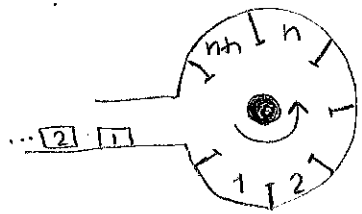
} 5 classes ← Catalan #. Why?

16 parking fns

Theorem There are $(n+1)^{n-1}$ parking functions of length n

Proof

Put them in a "rompoi" with n+1 spots instead:



- If a car can't find a spot, it keeps circling around
- Now n+1 is a possible preferred spot

$a=(4,3,3,4,1) \to$ 

Now all cars can park and exactly one spot is left empty. Note:

- if $(a_1, a_2, \dots, a_n)$ leaves i empty, $(a_1+1, a_2+1, \dots, a_n+1)$ leaves $i+1$ empty (modulo $n+1$)
- then there are $\frac{1}{n+1} \cdot (n+1)^n$ preference fns that leave $n+1$ empty
- these are precisely the parking functions! ∎

Q Bijection?

(parking fns. of size n) $\longleftrightarrow$ (trees on [n+1])

- Think
- Google