Compositional Formula

Lecture 14
10.22.13

If B, C are labeled combin classes with $B_0 = \emptyset$, let

$$\boxed{B \circ C = \sum_{k\geq 0}^{\infty} B_k \times Set_k(C)}$$

- partition set into $\{S_1, \dots, S_k\}$
- put a C structure on each S_i
- put a B structure on the set $\{S_1, \dots, S_k\}$

Then $\boxed{B \circ C(z) = B(C(z))}$

Pf $B(C(z)) = \sum_{k\geq 1} b_k \frac{C(z)^k}{k!} = \sum_{k\geq 1} b_k (Set_k C)(z)$ ∎

Ex

Ordered set partitions $[n] = S_1 \sqcup \dots \sqcup S_k$

$$OrdSetPartitions = Perm \circ Set_{\geq 1}$$

$$\sum_{n\geq 0} o(n) \frac{x^n}{n!} = \frac{1}{1-[e^x-1]} = \frac{1}{2-e^x}$$

We can introduce a weight $y^{\#parts}$ by introducing a weight of y to each set in $Set_{\geq 1}$: $yOrdSetPart = Perm \circ ySet_{\geq 1}$

$$\sum_{\substack{n\geq 0\\ k\geq 0}} o(n,k) \frac{x^n}{n!} y^k = \frac{1}{1-(yx+y\frac{x^2}{2!}+y\frac{x^3}{3!}+\dots)} = \frac{1}{1+y-ye^x}$$

Ex. (HW2)

$f(n,k) = \#$ of seqs $\emptyset = S_0, S_1, \dots, S_n = \emptyset$ (each $S_i \subseteq [k]$) such that for each i,
- $S_i \supset S_{i+1}$ and $|S_i - S_{i+1}| = 1$, or
- $S_i \subset S_{i+1}$ and $|S_{i+1} - S_i| = 1$

Ex: $\emptyset, 1, 12, 1, \emptyset, 3, 13, 135, 35, 5, 25, 2, \emptyset$ $\quad n=4,\ k=12$

$\downarrow$

1 2 2 1 3 1 5 1 3 2 5 2

$\downarrow$

$1,4,6,8 \mid 2,3,10,12 \mid 5,9 \mid \emptyset \mid 7,11$

a weak partition of n into k even parts

So

This $= Perm \circ yEvenSet$

where

$$yEvenSet_{\geq 1} = y\frac{x^0}{0!} + y\frac{x^2}{2!} + y\frac{x^4}{4!} + \dots = y\left(\frac{e^x+e^{-x}}{2}\right) = y\cosh x$$

So

$$\sum_{n,k\geq 0} f(n,k)\frac{x^n}{n!}y^k = \frac{1}{1-y\cosh x} = \frac{1}{1-y\left(\frac{e^x+e^{-x}}{2}\right)}$$

$$= \sum_{k\geq 0} y^k \left(\frac{e^x+e^{-x}}{2}\right)^k$$

$$= \sum_{k\geq 0} \frac{y^k}{2^k} \sum_{i=0}^{k} \binom{k}{i} e^{x(k-i)} (e^{-x})^i$$

$$= \sum_{k\geq 0} y^k \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} \sum_{n\geq 0} \frac{(k-2i)^n}{n!} x^n$$

$$f(n,k) = \frac{1}{2^k} \sum_{i=0}^{k} \binom{k}{i} (k-2i)^n$$

Derivatives:

If $F(x)$ is the GF for f-structures on $[n]$

then $F'(x)$ is the GF for f-structures on $[n+1]$.

Pf $F'(x) = \left(\sum_{n\geq 0} f(n)\frac{x^n}{n!}\right)' = \sum_{n\geq 1} f(n)\frac{x^{n-1}}{(n-1)!} = \sum_{n\geq 0} f(n+1)\frac{x^n}{n!}$.

Ex

Permutations:

$n=8$; Perm$'$ $\qquad$ Perm $*$ Perm

137295846 $\longleftrightarrow$ (1372, 5846)

$f'(x) = f(x)^2 \qquad f(0)=1 \qquad$ a differential equation!

$-\frac{f'}{f^2} = -1$

$(1/f)' = -1$

$1/f = -x + c$

$f(x) = \frac{1}{c-x} \quad \Rightarrow f(x) = \frac{1}{1-x} = \underset{\uparrow}{\sum_{n\geq 0} n!}\,\frac{x^n}{n!}$

perm of $[n]$

Ex.

Alternating permutation: $w_1 < w_2 > w_3 < w_4 \cdots < w_{n-1} > w_n$

(only possible if n is odd)

Let E_n = # of alt perms of $[n]$ $\qquad$ (n odd)

$n=8$ (even) $\qquad$ Alt$'$ $\qquad$ Alt $*$ Alt

253948176 $\longleftrightarrow$ (253, 48176)

$\neq\emptyset$ odd $\qquad \neq\emptyset$ odd

This works for $n\geq 2$ (not $n=0$) so

$$E'(x) = E(x)^2 + 1 \qquad E(0)=0$$

$$E(x) = \tan x = x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + \cdots$$

Similarly, the egf for alternating permutation of even length is

$$E_{even}(x) = \sec x = 1 + \frac{x^2}{2!} + 5\frac{x^4}{4!} + \cdots$$

These numbers $1,1,1,2,5,16,61,\ldots$ are the Euler numbers.

Note: We get combin. interp. of • $1+\tan^2 x = \sec^2 x$

"Combinatorial trigonometry" $\qquad \sum_k E_{2k}E_{2n-2k} = \sum_k E_{2k+1}E_{2n-2k-1}$

• $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

Rooted structures

If $F(x)$ is the EGF for f-structures on $[n]$,

$xF'(x)$ is the EGF for rooted f-structures on $[n]$.

Ex. $T(x)$ = trees

$xT'(x)$ = rooted trees