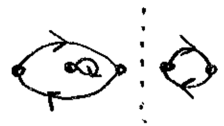
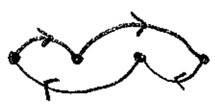
Ex: a_n = # of permutations of $[n]$ such that $|\pi(i)-i| \le 2$ for all i.

Ex: $\pi = 3\,2\,1\,5\,4 \longrightarrow$ 

irreducible ones:

 and reverse 

 and reverse 

 and reverse

 and reverse

⋮

So

$$A(z) = \frac{1}{1-z-z^2-3z^3-3z^4-2z^5-2z^6-2z^7-\cdots}$$

$$= \frac{1-z}{1-2z-3z^3+z^5}$$

So $a_n = 2a_{n-2}+3a_{n-3}-a_{n-5}$. Comb. proof?

The symbolic method for labeled structures and egfs

Lecture 13
10.15.13

Now we consider combinatorial structures of "size" n labeled by a set of n integers, usually $[n]$.

Let a_n = # of structures on $[n]$

$$A(z) = \sum_{n \ge 0} a_n \frac{z^n}{n!}$$ be the <u>exponential</u> GF.

Ex:
- permutations: $p_n = n!$ $\quad P(z) = \frac{1}{1-z}$
- sets: $u_n = 1$ $\quad U(z) = e^z$
- cycles: $c_n = (n-1)!$ $\quad C(z) = -\log(1-z)$
- "atomic class": $z_n = \begin{cases} 1 & n=1 \\ 0 & n \ne 1 \end{cases}$ $\quad Z(z) = z$

<u>Operations</u>

- $A+B = A \sqcup B$
- $A * B$ = { ways of partitioning $[n] = S \sqcup T$, choosing an A-structure on S, choosing a B-structure on T }
- $\mathrm{Seq}(B) = \varepsilon + B + (B*B) + (B*B*B) + \cdots$
  $= \bigcup_{k \ge 0}^{\infty} \mathrm{Seq}_k(B)$ = { ways of choosing an ordered set partition $[n] = \bigsqcup_{i=1}^{k} S_i$, choosing a B-structure on each S_i }
- $\mathrm{Set}(B)$ = { ways of choosing an <u>unordered</u> set partition $[n] = \bigsqcup_{i=1}^{k} S_i$, choosing a B-structure on each S_i }
  $= \bigcup_{k \ge 0}^{\infty} \mathrm{Set}_k(B)$

Exs:

ⓐ Perm = Set(Cycles) = Seq(Atom)

ⓑ Set Partitions = Set($\text{Set}_{\geq 1}$)

ⓒ Graphs = Set(Conn. Graphs)

Thm The exponential generating functions are:

$$(A+B)(z) = A(z) + B(z)$$

$$(A * B)(z) = A(z)B(z)$$

$$(\text{Seq}_k A)(z) = A(z)^k \qquad (\text{Seq } A)(z) = \frac{1}{1-A(z)}$$

$$(\text{Set}_k A)(z) = \frac{A(z)^k}{k!} \qquad (\text{Set } A)(z) = e^{A(z)}$$

Pf of second one:

If $A * B$ has c_n structures on $[n]$, then

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} = \text{coeff of } z^n \text{ in } \left(\sum_{k\geq 0} \frac{a_k}{k!} z^k\right)\left(\sum_{l\geq 0} \frac{a_l}{l!} z^l\right)$$

The others are easy corollaries. ■

Back to examples:

ⓐ $\frac{1}{1-z} = e^{-\log(1-z)} = \frac{1}{1-z}$

ⓑ $\sum_{n\geq 0} B_n \frac{z^n}{n!} = e^{e^z - 1}$ $\qquad \sum_{n\geq 0} S(n,k) \frac{z^n}{n!} = \frac{(e^z-1)^k}{k!}$

↑ Bell #s

ⓒ Here Graphs is the easy one!

$$g_n = 2^{\binom{n}{2}} \qquad cg_n = ?$$

Then

$$\sum_{n\geq 0} cg_n \frac{z^n}{n!} = \log \sum_{n\geq 0} \frac{2^{\binom{n}{2}} z^n}{n!}$$

Note: Radius of conv = 0. Still, an identity of formal power series.

Enumeration by connected components:

If a combin. object B decomposes uniquely into "connected"/"irred." objects A so that B = Set(A) then

$$B(z) = e^{A(z)}$$

$$\sum_{n\geq 0} \sum_{k\geq 0} b_k(n) y^k \frac{z^n}{n!} = e^{yA(z)} = B(z)^y$$

↳ B-objects on $[n]$ with k components

Pf. Exercise

Graphs:

$$\sum_{k,n} g_k(n) y^k \frac{z^n}{n!} = \left(\sum_{n\geq 0} \frac{2^{\binom{n}{2}} z^n}{n!}\right)^y$$

Permutations:

$$\sum_{k,n} c(n,k) y^k \frac{z^n}{n!} = \left(\frac{1}{1-z}\right)^y = (1-z)^{-y}$$

$$= \sum_{n\geq 0} \binom{-y}{n} (-z)^n$$

$$= \sum_{n\geq 0} y(y+1)\cdots(y+n-1) \frac{z^n}{n!}$$

↑ Signless Stirling #s of first kind

Ex: Involutions

Let i_n be the number of permutations π of $[n]$ such that $\pi^{-1} = \pi$.

Note: $\pi^{-1} = \pi \iff \pi^2 = id \iff$ all cycles of π have length 1 or 2

So

$$\sum_{n\geq 0} i_n \frac{z^n}{n!} = e^{z + \frac{z^2}{2}}$$

Ex Derangements

Let d_n be the number of derangements π of $[n]$; that is, permutations with $\pi(i) \neq i$ for all i. Then

$$\sum_{n\geq 0} d_n \frac{z^n}{n!} = e^{\frac{z^2}{2} + \frac{z^3}{3} + \cdots} = e^{-\log(1-z) - z} = \frac{e^{-z}}{1-z}$$

$$= e^{-z} + ze^{-z} + z^2 e^{-z} + \cdots$$

so

$$\frac{d_n}{n!} = \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^{n-2}}{(n-2)!} + \cdots + \frac{(-1)^0}{0!}$$

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right) \sim \frac{n!}{e}$$

So the probability that a permutation has no fixed points is $\approx 1/e$ ("Matching problem"/"Hat problem" in probability)

Revisiting the cycle indicator of S_n:

The type of $w \in S_n$ is $(c_1, \ldots, c_n)$ where c_i is the number of cycles of length i. Let $t^{type(w)} = t_1^{c_1} \cdots t_n^{c_n}$. Let

$$Z_n = \frac{1}{n!} \sum_{w \in S_n} t^{type(w)}$$

$$Z_3 = \frac{1}{6}(t_1^3 + 3t_1 t_2 + 2t_3)$$

Then

$$\sum_{n\geq 0} Z_n x^n = e^{t_1 x + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots}$$