Combinatorial meaning of operations on ordinary generating functions

lecture 12
10.10.13

A "combinatorial class" is a set A with a size function $|\cdot|: A \to \mathbb{N}$ such that the number of elements of size n is finite for all n.

Let a_n = # of elts of size n

$$A(x) = \sum_{n=0}^{\infty} a_n z^n$$

Ex: $W_{\{0,1\}} = \{\text{binary words}\} = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$

$|w|$ = length of w

$$W(x) = \sum_{n\geq 0} 2^n z^n \quad \frac{1}{1-2z}$$

Ex: $\epsilon = \{\phi\}$ | $|\phi| = 0$ | GF = 1

$\mathcal{Z} = \{\phi\}$ | $|\phi| = 1$ | GF = z

Operations

(+) $A + B = A \sqcup B$

(×) $A \times B = \{\alpha\beta : \alpha \in A, \beta \in B\}$ $\quad |\alpha\beta| = |\alpha| + |\beta|$

(Seq) $\mathrm{Seq}(A) = \varepsilon + A + (A \times A) + (A \times A \times A) + \cdots$

(Need: no elt of A has size 0.)

Exs

- $W_{\{0,1\}} = \mathrm{Seq}(\{0,1\})$ where $|0| = |1| = 1$
- compositions:

  $\mathrm{Comp} = \mathrm{Seq}(\{1,2,3,\dots\})$ where $|k| = k$ $(k \in \mathbb{N})$

  $|232141| = 12$
- multisets of $[m]$:

  $\text{Multisubsets of } [m] = \mathrm{Seq}(\{1\}) \times \cdots \times \mathrm{Seq}(\{m\})$

  $|11344445| = 7$ where $|k| = 1$ $(k = 1, \dots, m)$
- partitions with parts $\leq m$:

  $\text{Partitions}_{\leq m} = \mathrm{Seq}(\{1\}) \times \cdots \times \mathrm{Seq}(\{m\})$ where $|k| = k$

  $|5444311| = 22$

Thm The ordinary generating fns. are given by

$$(A+B)(z) = A(z) + B(z)$$
$$(A \times B)(z) = A(z)B(z)$$
$$(\mathrm{Seq}(A))(z) = \frac{1}{1 - A(z)}.$$

Remark: In the field of formal power series $\mathbb{C}[[z]]$

$a_0 + a_1 z + a_2 z^2 + \cdots$ is invertible $\Leftrightarrow a_0 \neq 0$.

So in our example:

- $W_{\{0,1\}}(z) = \dfrac{1}{1-(z+z)} = \dfrac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$

- $\text{Comp}(z) = \dfrac{1}{1-(z+z^2+z^3+\cdots)} = \dfrac{1}{1-\frac{z}{1-z}} = \dfrac{1-z}{1-2z}$

  $\sum_{n\geq 0} c(n) z^n = \sum_{n\geq 0} 2^{n-1} z^n$

- $(\text{Multisets of } [m])(z) = \dfrac{1}{1-z}\cdots\dfrac{1}{1-z} = (1-z)^{-m}$

  $\sum_{n\geq 0} \left(\!\binom{m}{n}\!\right) z^n = \sum_{n\geq 0} (-1)^n \binom{-m}{n} z^n$

- $(\text{Partitions}_{\leq m})(z) = \dfrac{1}{1-z}\cdot\dfrac{1}{1-z^2}\cdots\dfrac{1}{1-z^m}$

  $\sum_{n\geq 0} P_{\leq m}(n) z^n$

(Exercise: Prove, similarly, that $\sum_{n\geq k} S(n,k) x^n = \dfrac{x}{1-x}\cdot\dfrac{x}{1-2x}\cdots\dfrac{x}{1-kx}$ (See HW4))

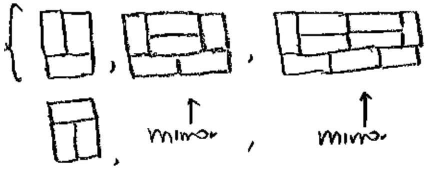
More interesting:

Ex. $T = \{\text{domino tilings of } 2\times n \text{ rectangles}\}$ ← $|\cdot| = n$

$T_{irred} = \{$"irred" domino tilings of $2\times n$ rectangles which have no "fault lines"$\} = \{ ▯, ⊟ \}$

$T = \text{Seq}(T_{irred}) \Rightarrow \sum_{n\geq 0} t_n z^n = \dfrac{1}{1-z-z^2}$

Ex $U = \{\text{domino tilings of } 3\times n \text{ rectangles}\}$ ← $|\cdot| = n$

$U_{irred} = \{$  $, \ldots \}$

$\sum U_n z^n = \dfrac{1}{1-(2z^2+2z^4+2z^6+\cdots)} = \dfrac{1}{1-\frac{2z^2}{1-z^2}} = \dfrac{1-z^2}{1-3z^2}$

Remark: The number of domino tilings of a $2m\times 2n$ is:

$$4^{mn}\prod_{j=1}^{m}\prod_{k=1}^{n}\left(\cos^2\frac{j\pi}{2m+1} + \cos^2\frac{k\pi}{2n+1}\right)$$

(and similarly for even × odd).

This requires better techniques!

Ex. $D = \{\text{Dyck paths}\}$ $\qquad |\text{path}| = \frac{1}{2}(\#\text{steps})$

$I = \{$irred Dyck paths, that don't touch the x-axis$\}$

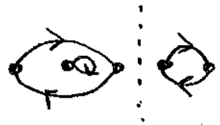
Clearly: $D = \text{Seq}(I)$

$I = \mathcal{E} * D$

So $D(z) = 1/_{1-I(z)}$ $\qquad I(z) = zD(z)$

$\Rightarrow D(z) = \dfrac{1}{1-zD(z)} \Rightarrow D(z) = \dfrac{1-\sqrt{1-4z}}{2z}$.

Ex: a_n = # of permutations of $[n]$ such that $|\pi(i)-i| \le 2$ for all i.

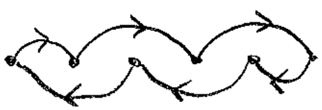
Ex: $\pi = 32154 \rightarrow$ 

Irreducible ones:

 and reverse 

 and reverse 

 and reverse

 and reverse

⋮

So

$$A(z) = \frac{1}{1-z-z^2-3z^3-3z^4-2z^5-2z^6-2z^7-\dots}$$

$$= \frac{1-z}{1-2z-3z^3+z^5}$$

So $a_n = 2a_{n-2} + 3a_{n-3} - a_{n-5}$. Comb. proof?