Remark

Since $\prod_{k\geq 1} \frac{1}{1-x^k} = \sum_{n\geq 0} p(n)x^n$

$$\prod_{k\geq 1}(1-x^k) = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\dots$$

when we multiply them and compare coeffs of x^n we get

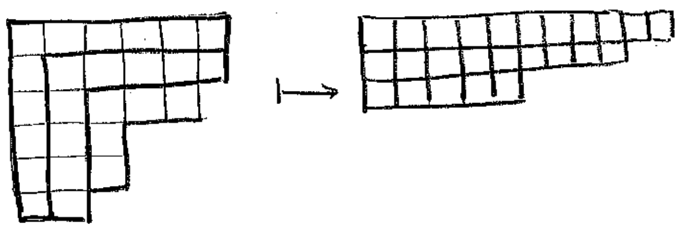
$$0 = p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-p(n-12)\cdots$$

This recurrence is the best way to date to compute $p(1), p(2), \dots$. There are other ways of computing $p(n)$ only. Also $p(n) \sim e^{\pi\sqrt{2n/3}}/4\sqrt{3}\,n$

(Compare with $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$)

Prop The # of self-conjugate partitions of n equals the # of partitions of n into odd parts

Pf.



$6+6+5+3+3+2 \longmapsto 11+9+5$ ∎

Formal Power Series

Now that we've played enough with formal power series to know what we might need to worry about, let's discuss why we don't need to worry.

Let R = comm. ring. (For us usually $R=\mathbb{R}$ or $\mathbb{C}$)

A __formal power series__ is a sequence

$(a_0, a_1, a_2, \dots)$ which we write "$a_0+a_1x+a_2x^2+\dots = A(x)$"

$(a_i \in R)$

Write $a_n = [x^n]A(x)$

The ring of formal power series $R[[x]]$ has ops

$+: (a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{Z}} = (a_n+b_n)_{n\in\mathbb{Z}}$

$\cdot: (a_n)_{n\in\mathbb{N}} \cdot (b_n)_{n\in\mathbb{Z}} = (a_0b_n + a_1b_{n-1}+\dots+a_nb_0)_{n\in\mathbb{Z}}$

(Consistent with our power series notation)

We have $0 = (0,0,\dots)$, $1 = (1,0,0,\dots)$

Easy: assoc. of $+$, of $\cdot$

comm. of $+$, of $\cdot$

distrib. of $+, \cdot$

See: EC1, Sec 1.1

Ivan Niven "Formal Power Series" (Amer Math Monthly)

There is a distinction between formal and analytic power series, but:

> **Principle:** Any identity of power series which
> - holds analytically for small enough $|z|$
> - makes sense for formal power series
>
> also holds in the ring of formal power series.

This is clearer through some examples.

Ex. 1. $\sum_{n\geq 0} r^n x^n = \frac{1}{1-rx}$

Here this means $\left(\sum_{n\geq 0} r^n x^n\right)(1-rx) = 1$ as formal power series. But

$$[x^n]\,LHS = \begin{cases} r^n + r^{n-1}(-r) = 0 & n\geq 1 \\ 1 & n=0 \end{cases} \quad \checkmark$$

Ex 2 $\sum_{n\geq 0} x^n/n! \sum_{n\geq 0} (-x)^n/n! = 1$

Alg. This makes sense in $\mathbb{R}[[x]]$; it says

$$\sum_{k=0}^{n} \frac{1}{k!}\frac{(-1)^{n-k}}{(n-k)!} = \begin{cases} 0 & n\geq 1 \\ 1 & n=0 \end{cases}$$

This follows from $(1+-1)^n = \begin{cases} 1 & n\geq 1 \\ 0 & n=0 \end{cases}$

An. We can also invoke analysis; this says

$$e^x \cdot e^{-x} = 1$$

which is true for all $x \in \mathbb{C}$. Then just use:

> **Thm:** If two power series represent the same function in a neighborhood of 0, their coeffs are equal.

Ex 3 (A non-example)

The analytic identity $e^{x+1} = e \cdot e^x$ does not give an identity in $\mathbb{R}[[x]]$, because

$$\sum_{n\geq 0} (x+1)^n/n! \quad \text{is not a formal power series.}$$

The coeff. of x^N has infinitely many contributions!

To make sense of (some) infinite sums, need to define convergence in $\mathbb{R}[[x]]$.

Say $F_1(x), F_2(x), \ldots \to F(x)$

if for any n there exists N such that

$$[x^n]F_N(x) = [x^n]F_{N+1}(x) = \ldots = [x^n]F(x).$$

Let $\deg F(x) = \min n$ s.t. $[x^n]F(x) \neq 0$

> **Prop** $\sum_{j=0}^{\infty} A_j(x)$ converges $\iff \lim_{j\to\infty} \deg A_j(x) = \infty$

So: $\sum_{n\geq 0} \frac{(x+1)^n}{n!}$ doesn't, $\sum_{n\geq 0} \frac{[x A(x)]^n}{n!}$ does.

So: If $F(x), G(x) \in R[[x]]$, we can define

$$F(G(x)) = \sum_{n\geq 0} f_n \Big(\sum_{m\geq 0} g_m x^m\Big)^n \quad \text{iff} \quad g_0 = G(0) = 0.$$

$$\boxed{\text{Prop. } \prod_{j\geq 0}(1+A_j(x)) \text{ converges} \iff \lim_{j\to\infty} \deg A_j(x) = \infty \quad (A_j(0)=0)}$$

Ex 4 We showed

$$\sum_{n\geq 0} P_{\leq k}(n) x^n = \prod_{i=1}^{k} \frac{1}{1-x^i} \qquad (1)$$

and concluded

$$\sum_{n\geq 0} P(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i} = \prod_{i=1}^{\infty}(1+\underbrace{x^i+x^{2i}+\ldots}_{\deg = i}) \qquad (2)$$

The RHS is defined and what we are doing is taking the limit of (1) as $k\to\infty$; the coeffs stabilize to those of (2).

$\boxed{\text{Prop } R[[x]] \text{ is an integral domain if } R \text{ is an integral domain}}$ $(ab=0 \Rightarrow a=0 \text{ or } b=0)$

Pf If $A(x)B(x)=0$ where $A(x) = \overset{\neq 0}{a_m} x^m + \ldots$

$B(x) = \underset{\neq 0}{b_n} x^n + \ldots$

then $A(x)B(x) = \underset{\neq 0}{a_m b_n} x^{m+n} + \ldots$ ■

Ex 5 The Catalan GF satisfied

$$x\,C(x)^2 = C(x) - 1$$

We did

$$4x^2 C(x)^2 - 4x C(x) + 1 = 1 - 4x$$

$$[1 - 2x C(x)]^2 = 1 - 4x$$

Now:

$$(1+x)^r := \sum_{n\geq 0} \binom{r}{n} x^n$$

satisfies

$$\left[(1-4x)^{1/2}\right]^2 = 1 - 4x$$

since it satisfies it for $|x| < 1/4$. Then

$$[1-2x C(x)]^2 = \left[(1-4x)^{1/2}\right]^2$$

Now, $A^2 = B^2 \Rightarrow (A-B)(A+B) = 0 \Rightarrow A = \pm B$.

Since they both have $[x^0] = 1$,

$$1 - 2x C(x) = (1-4x)^{1/2}$$

"Calculus": Define $\left(\sum_{n\geq 0} f_n x^n\right)' = \sum_{n\geq 0} (n+1) f_{n+1} x^n$

We have $(FG)' = F'G + FG'$

$(F(G(x)))' = F'(G(x))G'(x) \qquad (G(0)=0)$

$\vdots$

So, e.g. if $G'(x) = F'(x)/F(x) \qquad G(0) = 0 \qquad F(0) = 1$

$G'(x) = [\log F(x)]'$

$G(x) = \log F(x) \longrightarrow F(x) = e^{G(x)}$