Set partitions

Lecture 10
10.08.13

A *set partition* of a set S is a collection $\Pi = \{B_1, \dots, B_k\}$ of non-empty blocks $B_i \subseteq S$ with

$$S = B_1 \sqcup \dots \sqcup B_k \qquad \text{that is, } S = B_1 \cup \dots \cup B_k, \ B_i \cap B_j = \emptyset \text{ for } i \neq j.$$

Let

$S(n,k) = \#$ of partitions of $[n]$ into k blocks

be the Stirling numbers of the second kind. Let

$B_n = \#$ of partitions of $[n]$

be the Bell numbers.

Prop $S(n,k) = kS(n-1,k) + S(n-1,k-1)$

Pf To make such a partition:

- Add n to a block of a partition of $[n-1]$ into k blocks, or
- Add a singleton block $\{n\}$ to one of $[n-1]$ into $k-1$ blocks. ∎

Prop
$$\sum_{n=k}^{\infty} S(n,k)\, x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$
$$\sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$$

Pf. Induct. (There are nicer proofs, we might discuss them later)

Prop $$x^n = \sum_{k=0}^{n} S(n,k)[x(x-1)\cdots(x-k+1)] \qquad (1)$$

Pf. It's enough to prove it for $x \in \mathbb{N}$.

\# of ways of putting balls $1, \dots, n$ into boxes $1, \dots, x$:

- $\underset{\text{ball } 1}{x} \cdot \dots \cdot \underset{\text{ball } n}{x} = x^n$ ways
- Assume we use exactly k boxes
  - $\binom{x}{k}$ choices for the boxes
  - $S(n,k)$ choices for the k groups of balls $B_1, \dots, B_k$ going in the k boxes
  - $k!$ choices for which groups go in which boxes

$\Rightarrow \sum_k \binom{x}{k} S(n,k) k! = RHS$ ∎

Recall: The Stirling #s of 1st kind are $s(n,k) = (-1)^{n-k} c(n,k)$, and
$$\sum_{k=0}^{n} c(n,k) x^k = x(x+1)\cdots(x+n-1)$$
so
$$\boxed{x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)\, x^k} \qquad (2)$$

Therefore
$B_1 = \{x^0, x^1, x^2, \dots\}$ and $B_2 = \{1, x, x(x-1), x(x-1)(x-2), \dots\}$
are bases for the vector space $\mathbb{R}[x]$, and the transition matrices from B_1 to B_2 and back are
$$(S(n,k))_{n,k=0}^{\infty}, \quad (s(n,k))_{n,k=0}^{\infty} \quad \text{(lower } \Delta\text{, inverses)}$$

Balls into boxes

(Rota's "twelvefold way" - putting it all together)

<u>Question</u>: How many ways are there to put n balls into m boxes?

<u>Answer</u>: It depends.:

- Four variants on balls/boxes | are the boxes distinguishable? are the balls?
- Three variants on placement | is the placement "f: balls $\to$ boxes" arbitrary? injective? surjective?

This gives twelve questions.

We just did 1, 2, 3.

We also get 9, then 7. 8 is clear.

12: Partition our n indist. balls into m indist. groups. This is just a partition of n into m parts $\Rightarrow P_m(n)$

10: Now partition into $\leq m$ parts

11 is clear

5: Out of the m boxes, which n contain one ball?

6: If box i contains a_i balls, then $n = a_1 + \cdots + a_m$ is a composition of n into m parts

4: Now $n = a_1 + \cdots + a_m$ is a weak composition, and there are $\left(\!\binom{m}{n}\!\right) = \binom{m+n-1}{n}$ of them

n balls	m boxes	any placement	$\leq$ one ball per box	use all boxes
dist.	dist.	m^n	2 $m(m-1)\cdots(m-n+1)$	3 $S(n,m)\cdot m!$
indist.	dist.	4 $\left(\!\binom{m}{n}\!\right)$	5 $\binom{m}{n}$	6 $\binom{n-1}{m-1}$
dist.	indist.	7 $S(n,1)+\cdots+S(n,m)$	8 $\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	9 $S(n,m)$
indist	indist	10 $P_{\leq m}(n)$	11 $\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	12 $P_m(n)$