Enumerative Combinatorics

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Enumerative Combinatorics

Lecture 1
8·29·13

① Enumerative combinatorics is about counting

Main problem: To count the number of objects with some given properties

② Enumerative combinatorics is not just about counting.

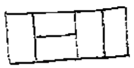
Basic fact: To count these objects we usually need to understand them first.

We'll study combinatorial structure.

Most often, we wish to count the objects in a set S_n for $n = 0, 1, 2, \dots$. Let $a_n = |S_n|$.

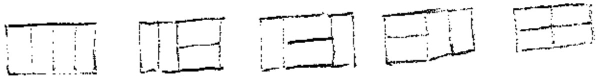
Example

Let t_n be the number of ways of tiling a $2 \times n$ rectangle with 2×1 "dominoes".



(One tiling for $n=5$)

Ex: $n=4$:



$t_4 = 5$

Question: "Compute a_n".

What is a satisfactory answer?

4 common kinds: ① Explicit formula

② Recurrence formula

③ Formula for generating function

④ Asymptotic formula

In the tiling problem:

② Recurrence: Express t_n in terms of $t_{n-1}, t_{n-2}, \dots, t_0$

Common trick: Study what happens to the "first" piece of the object.

Top left corner must be covered  or 

If vertically: the remaining $2 \times (n-1)$ rectangle has t_{n-1} tilings

If horizontally: must start with two horizontals, then the $2 \times (n-2)$ rectangle has t_{n-2} tilings.

$$\boxed{t_n = t_{n-1} + t_{n-2} \quad (n \geq 2), \quad t_1 = 1, \quad t_2 = 2}$$

$1, 2, 3, 5, 8, 13, 21, \dots$

So $b_n = f_{n+1}$ in terms of the Fibonacci sequence

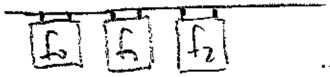
$$\boxed{f_n = f_{n-1} + f_{n-2} \quad (n \geq 2), \quad f_0 = 0,\ f_1 = 1}$$

This recurrence allows us to compute $a_1, a_2, a_3, \dots$ without having to draw the tilings.

③ The generating function for this sequence is

$$F(x) = f_0 + f_1 x + f_2 x^2 + \cdots = \sum_{n \geq 0} f_n x^n$$

You could regard this as an analytic function.
We regard it as a "formal power series":
a clothesline:



where we can perform the usual alg. operations.
(More on this later.)

$$\begin{aligned} F(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots \\ &= 0 + x + \boxed{f_1 x^2} + \boxed{f_2 x^3} + \boxed{f_3 x^4} + \cdots \\ &\qquad\quad + \boxed{f_0 x^2} + \boxed{f_1 x^3} + \boxed{f_2 x^4} + \cdots \\ &= x + \boxed{x F(x)} \\ &\qquad + \boxed{x^2 F(x)} \end{aligned}$$

so $\boxed{F(x) = \dfrac{x}{1-x-x^2}}$

This is often an excellent answer, because it allows us to derive ①, ②, and ④:

① Using partial fractions,

$$F(x) = \frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

where $1-x-x^2 = (1-\alpha x)(1-\beta x)$. After some work,
We get $\alpha = \frac{1+\sqrt{5}}{2}$ $\beta = \frac{1-\sqrt{5}}{2}$ $A = \frac{1}{\sqrt{5}}$ $B = -\frac{1}{\sqrt{5}}$

So

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right) \\ &= \frac{1}{\sqrt{5}}\left[\sum_{n\geq 0} \alpha^n x^n - \sum_{n \geq 0} \beta^n x^n\right] \\ &= \sum_{n \geq 0} \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) x^n \end{aligned}$$

which gives

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}$$

This is an explicit formula, good!

But to compute f_{20} by hand, the recurrence is better.

④ An asymptotic formula should tell us "about how big a_n is" — how quickly does it grow with n? Linearly? Polynomially? Exponentially?

In this case it's easy because $\left|\frac{\sqrt{5}-1}{2}\right| < 1$ so $\left(\frac{\sqrt{5}-1}{2}\right)^n \to 0$ as $n \to \infty$. Therefore

$$\boxed{f_n \sim \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n}$$

grows exponentially.

Note: $f_n \sim g_n$ means $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$.

Often, an exact formula ① gives no insight on the asymptotic behavior ④.

But complex analysis often makes it straightforward to go from the gen. fn ③ to the asymptotic behavior.

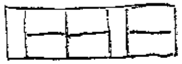
<u>Note</u>: We won't study this in this course

See H. Wilf: generatingfunctionology.

<u>Structure</u>: Along the way we see that these tilings are structurally very simple: they are just sequences of ▯ and ⊟.

Bijection:

$$\left(\begin{array}{c}\text{domino tilings of}\\ \text{a } 2\times n \text{ rectangle}\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{ways of writing } n \text{ as}\\ \text{a sum of 1s and 2s}\end{array}\right)$$

 $\longleftrightarrow$ $8 = 1+2+2+1+2$

In fact we will see that

$$\left(\begin{array}{c}\text{domino}\\ \text{tilings}\\ \text{of } 2\times n\text{'s}\end{array}\right) \text{"="} \frac{1}{1-▯-⊟} \longleftrightarrow F(x) = \frac{x}{1-x-x^2}$$

(Remark: There are often several different formulas for the same amount. For example,

$$f_n = \sum_{k=0}^{n-1} \binom{n-k-1}{k}$$
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