federico ardila
san francisco state university
universidad de los andes

homework four . due **THURSDAY**[1] oct 17 at 11:59pm in your local time

Note. You are highly encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend that you put away the notes from your discussions with others, and make sure that you can reproduce the solutions by yourself.)
You must turn in your homework – **in one .pdf file** by email at `icountslowly@gmail.com`.

(As in "real" mathematics, the questions get more vague as we progress. Try to find the most interesting interpretations to these open-ended questions.)

1. (Dull sequences) A sequence of positive integers is *dull* if for any $k > 1$ which appears in the sequence, the number $k-1$ appears at least once before the first occurrence of k.
   (a) Find the number of dull sequences of length n where the largest number is m.
   (b) (Bonus problem) Dull sequences are similar to the full sequences of Homework 1. You may have noticed that the number of full sequences of length n where the largest number is m is an Eulerian number. Are there other interesting variants of these questions?
2. (Another generating function for Stirling numbers) In class we considered the (ordinary and exponential) generating functions for Stirling numbers $S(n,k)$ when k is fixed and n varies. Now we consider the one where n is fixed and k varies: $p_n(x) = \sum_k S(n,k)x^k$.
   (a) Prove that $p_0(x), p_1(x), p_2(x), \ldots$ satisfies $p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)p_{n-k}(y)$.
   (b) (Bonus problem) You know at least one more sequence of polynomials satisfying (1). What can you say about such sequences?
3. (Another family of lattice paths) Let r_n be the number of paths from $(0,0)$ to $(2n,0)$ using steps $(1,1)$, $(1,-1)$, and $(2,0)$ which never go below the x axis.
   (a) Find the generating function[2] for r_n.
   (b) Prove that $(n+1)r_{n+1} = (6n-3)r_n - (n-2)r_{n-1}$ for all $n \geq 2$. This gives a **very** efficient method of computing these numbers.
4. (Tilings of a rectangle) Let t_n be the number of tilings of a $2 \times n$ rectangle into dominoes and L-shaped trominoes. Find the generating function[2] for t_n.
5. (What is combinatorics?) In at most half a page, react to one/some of these answers at: `http://www.math.ucla.edu/~pak/hidden/papers/Quotes/Combinatorics-quotes.htm`. .
6. (Bonus problem: Conway's sequence) Consider the sequence of words

$$1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, \ldots$$

   where each word is obtained by reading the previous word, saying the number of digits in each group of equal digits. For instance, the word 13112221 is read as "one one; one three; two ones; three twos; one one", giving rise to 1113213211, the next word in the sequence.
   (a) Prove that only the digits 1, 2, and 3 appear in the sequence.
   (b) Let a_n be the length of the nth word in the sequence. Prove that $a_1, a_2, a_3, \ldots$ satisfies a linear recurrence of order 71. Conclude that a_n grows exponentially with n.

[1] I hear the GRE is on Saturday...
[2] ordinary or exponential: your choice