enumerative combinatorics

federico ardila

san francisco state university
universidad de los andes

homework three . due friday oct 4 at 11:59pm in your local time

Note. You are highly encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend that you put away the notes from your discussions with others, and make sure that you can reproduce the solutions by yourself.)
You must turn in your homework – **in one .pdf file** by email at `icountslowly@gmail.com`.

Nota. Por la semana de receso, los estudiantes de Los Andes pueden entregar 3 problemas y obtener el puntaje completo. Si entregan más, obtendrán puntaje adicional.

1. (Two related (?) problems on permutations)
   (a) (Decorated permutations) A *decorated permutation* is a permutation where every fixed point i is given a sign $+$ or $-$. Prove that the number of decorated permutations of n is
   $$n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)\sim n!\,e.$$
   (b) (Average length of first run) A *run* in a permutation w is a maximal sequence of increasing consecutive elements $w_i < \ldots < w_{i+k}$. Prove that, among all permutations of $[n]$, the average length of the leftmost run is $\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\sim e-1$.
   (c) (Bonus. Are they related?) Explain bijectively the relationship between (a) and (b).
   (d) (Bonus. Average length of mth run) For fixed $1\le m\le n$, find the average length of the mth (from left to right) run among all permutations of $[n]$.
   (e) (Bonus. Average length of run) Find the average length of a run in a permutation of $[n]$.
2. (Avoiding patterns of length 3) Let w be any one of the six permutations of $[3]$. Prove that the number of permutations of $[n]$ avoiding the pattern w is the Catalan number C_n.
3. (A symmetric distribution for Dyck paths.) For a Dyck path P let $a(P)$ be the number of steps up that P takes before its first step down, and let $b(P)$ be the number of times that P returns to the x-axis after it leaves it for the first time. Prove that the statistics a and b are symmetrically distributed; that is:
   $$\sum_{P\text{ Dyck}} x^{a(P)}y^{b(P)} = \sum_{P\text{ Dyck}} x^{b(P)}y^{a(P)},$$
   where the sum is over all Dyck paths of length $2n$.
4. (Non-commutative binomial theorem.)
   (a) Let x,y,q be elements of a non-commutative ring such that $xy=qyx$, and q commutes with x and y. Prove that
   $$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}_q x^k y^{n-k}.$$
   (b) Generalize (a) to $x_1,\ldots,x_m,q$ such that $x_ix_j=qx_jx_i$ for $i\neq j$ and $x_iq=qx_i$ for all i.
5. (An identity for partitions) Prove that the number of partitions of n in which no part appears exactly once equals the number of partitions of n into parts not congruent to $\pm1 \pmod 6$.
6. (Bonus. A combinatorial proof of an analytic identity.) Give a combinatorial proof of the identity
   $$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{k=1}^{\infty}(1-q^{2k})(1+xq^{2k-1})(1+x^{-1}q^{2k-1}).$$