enumerative combinatorics

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homework two . due friday sep 20 at 11:59pm in your local time

Note. You are highly encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend that you put away the notes from your discussions with others, and make sure that you can reproduce the solutions by yourself.)
You must turn in your homework – **in one .pdf file** by email at `icountslowly@gmail.com`.

1. (Combinatorial identities.) Give combinatorial proofs for the following identities.
   (a) For any positive integers $n \geq k$,
   $$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$
   (b) For any positive integers $n \geq k$,
   $$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$
   (c) (Bonus problem.) For any positive integer n,
   $$\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{j=0}^{n} \binom{n}{j}\binom{2n-j}{n}(x-1)^j.$$
2. (Counting binary words by runs) A *binary word* is a word consisting of 0s and 1s. A *run* is a maximal string of consecutive 1s. For example the word 11010111011 has 4 runs. Find the number of binary words having exactly m 0s, n 1s, and k runs.
3. (Sequences of subsets)
   (a) Let $k, n \geq 1$ be given. Find the number of sequences $S_0, S_1, \ldots, S_k$ of subsets of $[n]$ such that for any $1 \leq i \leq k$ we have either:
   $$\begin{aligned} S_i \supset S_{i-1} \quad &\text{and} \quad |S_i - S_{i-1}| = 1, \text{ or} \\ S_i \subset S_{i-1} \quad &\text{and} \quad |S_{i-1} - S_i| = 1. \end{aligned}$$
   (b) Prove that there are exactly
   $$\frac{1}{2^n}\sum_{i=0}^{n} \binom{n}{i}(n-2i)^k$$
   such sequences with the additional property that $S_0 = S_k = \emptyset$.
4. (Permutations fixed by $\widehat{\ }$) Let $\widehat{\ } : S_n \to S_n$ be the fundamental transformation of S_n. Prove that the number of permutations w in S_n such that $\widehat{w} = w$ is the Fibonacci number f_{n+1}.
5. (Full sequences) A sequence of positive integers is *full* if for any k which appears in the sequence, the number $k-1$ appears at least once before the last occurrence of k. Find the number of full sequences of length n.
6. (Bonus problem: cycles of even and odd permutations.)
   (a) Let e_n be the total number of cycles among all even permutations of $[n]$, and o_n be the total number of cycles among all odd permutations of $[n]$. Prove that
   $$e_n - o_n = (-1)^n(n-2)!.$$
   (b) Give a bijective proof of (a).