enumerative combinatorics

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homework one . due friday sep 6 at 11:59pm in your local time

Note. You are highly encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend that you put away the notes from your discussions with others, and make sure that you can reproduce the solutions by yourself.)
You must turn in your homework – **in one .pdf file** by email at `icountslowly@gmail.com`.

1. (Numerical exercises.)
   (a) I forgot my code for the department's photocopier. I remember that it is 5 digits long, and it uses exactly three distinct digits between 0 and 9. Two of the digits are 1 and 7. How many different codes do I have to try (in the worst case) to recover it?
   (b) You want to walk from the point $(0,0)$ to the point $(8,8)$ in a square grid, travelling a total of exactly 16 unit steps. You must pass by point $(2,3)$ and you must avoid point $(6,5)$. How many different paths can you take?
   (c) A Millos fan had a quick glance at the table of positions of the Colombian soccer league, which has 18 teams. She can only remember a few things: Millos is first. Tolima is immediately above Santa Fé. Cali is higher than Junior, which is higher than Huila. Nacional is last. How many different tables of positions can there be?
2. (Words in a language.) Let w_n be the number of words of length n in the alphabet $\{A, B, C\}$ which do not have two consecutive consonants.
   (a) Write a recurrence relation for w_n.
   (b) Use (a) to compute the generating function $W(x) = w_0 + w_1x + w_2x^2 + \cdots$.
   (c) Use (b) to give an explicit formula for w_n.
   (d) (Bonus problem.) Give a combinatorial proof of (c).
3. (A generating function identity) Let k be a fixed positive integer. Prove the identity
$$\sum_{n_1,\ldots,n_k\geq 0} \min(n_1,\ldots,n_k)x_1^{n_1}\cdots x_k^{n_k} = \frac{x_1x_2\cdots x_k}{(1-x_1)\cdots(1-x_k)(1-x_1x_2\cdots x_k)}.$$
   where we are summing over all k-tuples of non-negative integers $n_1,\ldots,n_k$, and $\min(n_1,\ldots,n_k)$ denotes the smallest number among $n_1,\ldots,n_k$.
4. (Compositions of compositions) Let n be a fixed positive integer. Find the number of ways of choosing a composition α of n, and then choosing a composition of each part of α.
5. (Compositions with bounded parts) Let k be a fixed positive integer. For each n let $c_k(n)$ be the number of compositions of n such that every part is less than or equal to k. Prove that
$$\sum_{n\geq 0} c_k(n)x^n = \frac{1-x}{1-2x+x^k}.$$
6. (Bonus problem: NE-path tilings.) Consider an $n \times n$ square grid. An *NE-path* is a sequence of one or more unit squares such that each unit square is either directly above or directly to the right of the previous unit square. Find the number of ways of tiling the $n \times n$ square grid with NE–paths.

   A tiling of a 6×6 grid with NE–paths is shown on the right.