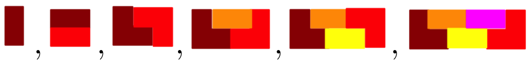
4. (Tilings of a rectangle) (Collaborators: Crista, Tahir) Let t_n be the number of tilings of a $2 \times n$ rectangle into dominoes and L-shaped trominoes. Find the generating function for t_n.

Proof. We want to find the generating function for t_n. Note that t_n is the number of tilings of a $2 \times n$ rectangle into dominoes and trominoes. So, we have a combinatorial class where our set T, with a size function $|\text{tilings}| = n$ (where n is the length), is defined below:

$$T = \{\text{domino and tromino tilings of rectangles of width } 2\},$$

and

$$T_{\text{irred}} = \{\text{Those without "fault line"}\}$$

$= \{$  $, \ldots \}.$

Note that these are the only irreducibles (including the "flips" of the $2 \times n$ rectangles for $n = 3, 4, \ldots$). The only way to have a 2×1 is to have a vertical domino. The only way to have a 2×2 is to have two horizontal dominoes. We cannot have a 2×2 starting with a vertical domino because we will have a fault line (hence, not irreducible). For a $2 \times n$ rectangle where $n = 3, 4, \ldots$, we must start and end with a tromino and only have horizontal dominoes in between to insure that we will not have any fault lines in our tiling. Now, $T = \text{Seq}(T_{\text{irred}})$ and our generating function is

$$\begin{aligned}
T(z) &= \frac{1}{1 - (z + z^2 + 2z^3 + 2z^4 + 2z^5 + 2z^6 + \cdots)} \\
&= \frac{1}{1 - z - z^2 - 2z^3\left(1 + z + z^2 + \cdots\right)} \\
&= \frac{1}{1 - z - z^2 - 2z^3\left(\frac{1}{1-z}\right)} \\
&= \frac{1}{1 - z - z^2 - \frac{2z^3}{1-z}} \\
&= \frac{1}{1 - z - z^2 - \frac{2z^3}{1-z}} \cdot \frac{1-z}{1-z} \\
&= \frac{1-z}{1 - z - z + z^2 - z^2 + z^3 - 2z^3} \\
&= \frac{1-z}{1 - 2z - z^3}.
\end{aligned}$$

□