3 | Another family of lattice paths

(a) Let r_n be the number of paths from $(0,0)$ to $(2n,0)$ using steps $(1,1)$, $(1,-1)$ or $(2,0)$ which never go below the x-axis. $r_0 = 1$ since the only path is the empty path. Now, if we want a path from $(0,0)$ to $(2(n+1),0)$ we can start with a $(2,0)$ step or with a $(1,1)$ step.

If we start with a $(2,0)$ step, the path from $(2,0)$ to $(2(n+2),0)$ is just another of these paths with length $2n$, so there are r_n options.

If we start with a $(1,1)$ step, let $k \geq 0$ be the minimum integer such that the path touches the x-axis for the first time after $(0,0)$ in $(2(k+1),0)$. Then the last step before reaching $(2(k+1),0)$ is a $(1,-1)$ and the path between $(1,1)$ and $(2k+1,1)$ is one of these paths of length $2k$, so there are r_k options. The path from $(2(k+1),0)$ to $(2(n+1),0)$ is also one of these paths and has length $2(n-k)$, so there are r_{n-k} options for this path. Note that k can be any number from 0 (if the second step is $(1,-1)$) to n (if the path only touches the x-axis at the beginning and the end).

Then we have the recurrence:

$$r_{n+1} = r_n + \sum_{k=0}^{n} r_k r_{n-k}$$

Let $G(x) = \sum_{n\geq 0} r_n x^n$ be the generating function for r_n. Multiplying by x^n on both sides of the previous equality and adding for $n \geq 0$ we get:

$$\begin{aligned}
\sum_{n\geq 0} r_{n+1}x^n &= \sum_{n\geq 0} r_n x^n + \sum_{n\geq 0}\left(\sum_{k=0}^{n} r_k r_{n-k}\right) x^n \\
\sum_{n\geq 0} r_{n+1}x^n &= G(x) + (G(x))^2 \\
\sum_{n\geq 1} r_n x^n &= xG(x) + x(G(x))^2 \\
(G(x) - r_0) &= xG(x) + x(G(x))^2 \\
0 &= x(G(x))^2 + (x-1)G(x) + 1 \\
0 &= (xG(x))^2 + x(x-1)G(x) + x \\
-x &= (xG(x))^2 + x(x-1)G(x) \\
-x + \frac{(x-1)^2}{4} &= (xG(x))^2 + x(x-1)G(x) + \frac{(x-1)^2}{4} \\
-x + \frac{x^2 - 2x + 1}{4} &= \left(xG(x) + \frac{x-1}{2}\right)^2 \\
\frac{x^2 - 6x + 1}{4} &= \left(xG(x) + \frac{x-1}{2}\right)^2 \\
\frac{\pm\sqrt{x^2 - 6x + 1}}{2} &= xG(x) + \frac{x-1}{2}
\end{aligned}$$

evaluating both sides in $x = 0$ we get $\frac{\pm 1}{2} = \frac{-1}{2}$, so the sign in the left side is a $-$. Then:

$$G(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}$$

(b) We want to prove that for $n \geq 2$:

$$(n+1)r_n = (6n-3)r_{n-1} - (n-2)r_{n-2}$$

This is equivalent to show that:

$$\sum_{n\geq 2}(n+1)r_n x^n = \sum_{n\geq 2}\left[(6n-3)r_{n-1} - (n-2)r_{n-2}\right] x^n$$

since two power series are equal if and only if their coefficients are equal. We have:

$$\begin{aligned}
\sum_{n\geq 2}(n+1)r_n x^n &= x\sum_{n\geq 2} n r_n x^{n-1} + \sum_{n\geq 2} r_n x^n \\
&= x\sum_{n\geq 2}(r_n x^n)' + \sum_{n\geq 2} r_n x^n \\
&= x(G(x) - 1 - 2x)' + (G(x) - 1 - 2x) \\
&= xG(x)' - 2x + G(x) - 1 - 2x \\
&= xG(x)' + G(x) - 4x - 1
\end{aligned}$$

$$\begin{aligned}
\sum_{n\geq 2}(6n-3)r_{n-1}x^n &= 6x^2\sum_{n\geq 2}(n-1)r_{n-1}x^{n-2}+3x\sum_{n\geq 2}r_{n-1}x^{n-1} \\
&= 6x^2\sum_{n\geq 2}(r_{n-1}x^{n-1})'+3x\sum_{n\geq 1}r_nx^n \\
&= 6x^2(G(x)-1)'+3x(G(x)-1) \\
&= 6x^2G(x)'+3xG(x)-3x
\end{aligned}$$

$$\begin{aligned}
\sum_{n\geq 2}(n-2)r_{n-2}x^n &= x^3\sum_{n\geq 2}(n-2)r_{n-2}x^{n-3} \\
&= x^3\sum_{n\geq 2}(r_{n-2}x^{n-2})' \\
&= x^3G(x)'
\end{aligned}$$

Then the equality we want to prove becomes:

$$xG(x)'+G(x)-4x-1=6x^2G(x)'+3xG(x)-3x-x^3G(x)'$$

or

$$(x^3-6x^2+x)G(x)'+(1-3x)G(x)-x-1=0$$

Note that

$$G(x)'=\frac{-\sqrt{x^2-6x+1}-3x+1}{2x^2\sqrt{x^2-6x+1}}$$

so

$$\begin{aligned}
& (x^3-6x^2+x)G(x)'+(1-3x)G(x)-x-1 \\
= & (x^3-6x^2+x)\frac{-\sqrt{x^2-6x+1}-3x+1}{2x^2\sqrt{x^2-6x+1}}+(1-3x)\frac{1-x-\sqrt{x^2-6x+1}}{2x}-x-1 \\
= & \frac{-(x^2-6x+1)-(3x-1)\sqrt{x^2-6x+1}}{2x}+\frac{(1-3x)(1-x)-(1-3x)\sqrt{x^2-6x+1}}{2x}-x-1 \\
= & \frac{-x^2+6x-1}{2x}+\frac{1-4x+3x^2}{2x}-x-1 \\
= & \frac{2x^2+2x}{2x}-x-1 \\
= & x+1-x-1 \\
= & 0
\end{aligned}$$

as we wanted to show. Then, for $n\geq 2$

$$(n+1)r_n=(6n-3)r_{n-1}-(n-2)r_{n-2}$$