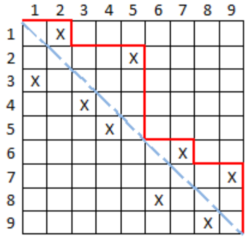
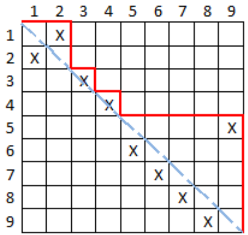
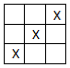
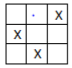
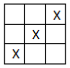
2. Recall that there exist a bijection ϕ from the permutations that avoid (321) in S_n to the set of dyck paths of length $2n$. The bijection consists of placing n $X's$ in a $n \times n$ board with coordinates (i, j), such that j is the value of the ith number in the permutation seen from left to right. The dyck path is constructed by walking to the right until the first X appears, after which I walk down until I reach a row with a X on the right, and the process is repeated. See image for better understanding.

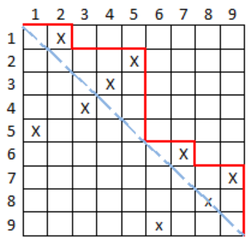
$(251347968) \mapsto$  $(213495678) \mapsto$ 

The construction of the inverse image is made by putting an X in the peaks of the path. These X's will produce an incomplete permutation that avoids (321). Since in most cases the board is still incomplete, I will complete the board by blocks. I consider a block to have as top right corner a X belonging to the peak of the path. Since I must avoid things like  the only possible way of doing this is by putting the X's in the $\searrow$ direction.

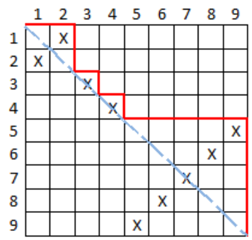
So the permutations in S_n that avoid (321) are in bijection with $2n$-length dyck paths. Note that there is a bijection between (321)-avoiding permutations and (123)-avoiding permutations in S_n. Consider the map $\Phi : S_n \to S_n$ defined by $(p_1 p_2 \dots p_n) \mapsto (p_n p_{n-1} \dots p_1)$. Φ is clearly a bijection, more importantly if $(p_1 ... p_n)$ avoids (321) then $(p_n ... p_1)$ avoids (123), otherwise there would be $i_1 < i_2 < i_3$ such that $p_{n-i_1} < p_{n-i_2} < p_{n-i_3}$ which is not possible because $n - i_3 < n - i_2 < n - i_1$ and $(p_1 ... p_n)$ avoided (321).

Therefore the permutations that avoid (123) are too in bijection with $2n$-length dyck paths.

Now consider the set of permutations in S_n that avoid (312). I may build a dyck path from any of these permutations the same way as ϕ does. Call this new map ψ. to construct the inverse image, make the same process of ϕ (inserting X's in the peaks of the path) but now the kind of configurations we don't like are like this  . So to guarantee that the path can be used to construct a unique permutation avoiding (312) each block must contain configurations of the form  only, because two distinct blocks have X's that are either all higher or all smaller than any X in the other block. Just like ϕ, there is only one way of doing it. See for example



$\xrightarrow{\psi^{-1}}$ (254317986)



$\xrightarrow{\psi^{-1}}$ (213498765)

Therefore (312) are in bijection with $2n$-length dyck paths under ψ. This implies that (213) is also in bijection with the paths under $\Phi \circ \psi$. It only remains to show that (132) and (231) are in bijection too.

Consider the bijection $\varphi : S_n \to S_n$ that maps $(p_1, \dots, p_n)$ into $(n - p_1, n - p_2, \dots, n - p_n)$. This is also a bijection since $n - p = n - q$ implies $p = q$. I claim that if $(p_1 ... p_n)$ avoids (312) then $\varphi(p_1 ... p_n)$ avoids (132). Suppose by contradiction that $(p_1 ... p_n)$ avoids (312), but $\varphi(p_1 ... p_n)$ does not avoids (132). Then there exists $i_1 < i_2 < i_3$ such that $n - p_{i_1} < n - p_{i_3} < n - p_{i_2} \Leftrightarrow p_{i_2} < p_{i_3} < p_{i_1}$, so $(p_1 ... p_n)$ contains a (312) and we obtain a contradiction. Therefore the permutations that avoid (132) are in bijection with those that avoid (312). Thus,

$$\{Dyck\ p_{2n}\} \xrightarrow{\phi^{-1}} S_n(321) \xrightarrow{\Phi} S_n(123)$$

$$\{Dyck\ p_{2n}\} \xrightarrow{\psi^{-1}} S_n(312) \xrightarrow{\Phi} S_n(213) \xrightarrow{\Phi^{-1} \circ \varphi} S_n(132) \xrightarrow{\Phi} S_n(312)$$

And the conclusion follows. $\square$