1. **(Two related problems on permutations)**

a) **(Decorated permutations)**

Let DP be the number of decorated permutations and DP_k be the number of decorated permutations with exactly k $-$'s. Then, summing over all possible k, we have by the sum principle that $DP = \sum_{k=0}^{n} DP_k$, since obviously no permutation is being included in two different DP_i and the sum exhausts the possibilities for the number of $-$'s . Furthermore, $DP_k = \binom{n}{k}(n-k)!$, where the combination counts the number of choices for the k elements with a $-$ sign, and the factorial gives the different permutations for the other $n-k$ elements. We don't worry about the elements with a $+$ sign because once the number of $-$'s is fixed, the other fixed elements of the permutation *have to* be assigned a $+$, thus there are no choices being excluded in the counting. Finally, summing up we obtain:

$$DP = \sum_{k=0}^{n} \binom{n}{k}(n-k)! = n! \sum_{k=0}^{n} \frac{1}{k!} \sim n!e$$

b) **(Average length of first run)**

For the first run to have length at least m, the first m elements of the permutation have to be ordered as $w_1 < w_2 < ... < w_m$; an equivalent set of conditions is $w_1 < w_2$ and $(w_1 < w_2) < w_3$ and ... and $(w_1 < ... < w_m - 1) < w_m$ where the parentheses are added for clarity of the argument. Note that the first condition is fulfilled with probability $1/2$, since when two numbers are picked up uniformly half the times the second will be larger than the first. Similarly, the third element will be greater than the first two with probability $1/3$ and the m condition will have probability $1/m$. Multiplying, we obtain that the first run will have length at least m with probability $q_m = 1/m!$.

The length of the first run will be exactly m if it is at least m but is not greater than m, thus, the probability is given by $p_m = q_m - q_{m+1} = 1/m! - 1/(m+1)!$ if $m < n$ and $p_n 1/n!$ otherwise. Let us call the average over all permutations El; it will be:

$$El = p_1 + 2p_2 + ... + np_n = (q_1 - q_2) + 2(q_2 - q_3) + ... + (n-1)(q_{n-1} - q_n) + nq_n =$$

$$= q_1 + q_2 + ... + q_n = \sum_{k=1}^{n} \frac{1}{k!} \sim e - 1$$

$c)$ **(average length of m run)**

Let q_{mk} be the probability that the first m runs have length at least k. We know that the number of permutations with exactly m runs is given by the Eulerian number $A(n, m)$ since a permutation with $m-1$ descents has m runs. This probability can be calculated by multiplying the number of permutations of $[k]$ with at most m runs with $\frac{1}{k!}$, which is the probability that a our set has length at least k. This implies:

$$q_{mk} = \frac{1}{k!}(A(k,1) + A(k,2) + ... + A(k,m))$$

Now, substract $q_{mk} - q_{m(k+1)}$ to obtain the probability that the first m runs have total length exactly k. From this we can obtain the average length of the first m runs, $L_{\leq m}$:

$$L_{\leq m} = (q_{m1} - q_{m2}) + 2(q_{m2} - q_{m3}) + 3(q_{m3} - q_{m2}) + ... =$$

$$= q_{m1} + q_{m2} + q_{m3} + ...$$

Based on this, the average length of the m run is:

$$L_{\leq m} - L_{\leq (m-1)} = (q_{m1} - q_{(m-1)1}) + (q_{m2} - q_{(m-1)2}) + (q_{m3} - q_{(m-1)3}) ... =$$

$$= (\frac{1}{1!}(A(1,1) + A(1,2) + ... + A(1,m)) - \frac{1}{1!}(A(1,1) + A(1,2) + ... + A(1,m-1))) +$$

$$(\frac{1}{2!}(A(2,1) + A(2,2) + ... + A(2,m)) - \frac{1}{1!}(A(2,1) + A(2,2) + ... + A(2,m-1))) + ... =$$

$$= \frac{1}{1!}A(1,m) + \frac{1}{2!}A(2,m) + ... = \sum_{k \geq 1} \frac{1}{k!} A(k,m)$$

Observe that this counts where done without taking n into account. Since n is fixed, the number there is a correlated limit for the number of runs and its lengths which has to be used when calculating $L_{\leq m}$. This limit will define the number of summands that are chosen in our result to calculate the average for the specific n.