Problem 5

The answer is $n!$, and we do a proof by induction. For $n = 1$, the only full sequence is 1. Suppose that we have already constructed the full sequences for $n - 1$. Call this set of full sequences F_{n-1} and assume that we know that $|F_{n-1}| = (n-1)!$. Now, we construct a bijection from $\{1, \ldots n\} \times F_{n-1}$ to F_n: given an i and a $(n-1)$-full sequence S, first we consider the slot where we are going to introduce the n-th element. If $i < n$, then this slot is: right before the i-th element of the S. If $i = n$, then the slot is simply after the whole sequence. In any case, having a fixed S and a slot, we calculate the maximum possible value that can be introduced in the slot so that the new sequence is still full. Note that this value is either the maximal element of the original S (we can always add this element) or that plus one (when the maximal appears before the slot). The resulting n-full sequence will be the image of the pair (i, S).

To prove that this function is a bijection, we note that it has an inverse. Given a n-full sequence R, find the first appearance from left to right of the maximal element that appears in R. Suppose that the position in which the maximal element m first appears is p. Then, the preimage of R is (p, S), where S is obtained from R by erasing that first appearance of m in R (or, in other words, erasing the p-th element of R). Also note that R has no other possible pre-image: the index is clearly one of the indices where m appears, but if we choose one which is not the first appearance then the maximum element that we can place in that slot is $m + 1$, not m, because there would be an m before the slot.