of such partitions. An expression for $f_k(n)$ could be

$$f_k(n) = \sum_{a_1+a_2+\cdots+a_n=k} \binom{k}{a_1, a_2, \cdots, a_n}$$

Where the sum is taken over all sequences $\{a_1, a_2, \cdots, a_n\}$ of even nonnegative even integers satisfying $a_1 + a_2 + \cdots + a_n = k$.

We will use an exponential generating function to obtain a

closed form for that sum. Consider the exponen
ing function

$$\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

The important observation is that

$$f_k(n) = \sum_{a_1+a_2+\cdots+a_n=k} \binom{k}{a_1, a_2, \cdots, a_n} = \left[\frac{x^k}{k!}\right] \left(\frac{e^x + e^{-x}}{2}\right)^n$$

i.e. the coefficient of $\frac{x^k}{k!}$ in the series $\left(\frac{e^x + e^{-x}}{2}\right)^n$

Expanding yields

$$\begin{aligned} \left(\frac{e^x + e^{-x}}{2}\right)^n &= \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \left(e^{-x}\right)^i \left(e^x\right)^{n-i} \\ &= \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} e^{x(n-2i)} \end{aligned}$$

And now it is clear that

$$\begin{aligned} f_k(n) &= \left[\frac{x^k}{k!}\right] \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} e^{x(n-2i)} \\ &= \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (n-2i)^k \end{aligned}$$