(3) (Sequences of subsets)

(a) Let $k, n \geq 1$ be given. Find the number of sequences $S_0, S_1, \ldots, S_k$ of subsets of $[n]$ such that for any $1 \leq n \leq k$ we have either:

$$S_i \supseteq S_{i-1} \quad \text{and} \quad |S_i - S_{i-1}| = 1, \text{ or}$$
$$S_i \subseteq S_{i-1} \quad \text{and} \quad |S_{i-1} - S_i| = 1.$$

First, as there are 2^n possible subsets there are 2^n possibilities for the set S_0. Now, with every transition from a subset S_i to S_{i+1} we "toggle" the membership of one of its elements. This means that for the transitions we are simply counting the number of sequences of length k that use numbers from $[n]$. Since there are n^k such sequences the answer is

$$2^n n^k.$$

(b) *Prove that there are exactly*

$$\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^k$$

such sequences with the additional property that $S_0 = S_k = \emptyset$.

For ease in notation, let

$$T_{k,n} = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^k.$$

Since every element that is "toggled" in must also be "toggled" out, we are simply counting the number of sequences of length k of numbers from $[n]$ that use each number an even number of times—for example, we can have

$$1\,2\,3\,2\,3\,1\,4\,4\,1\,4\,4\,1.$$

We will call such a sequence a *strongly even.*

Now, if k is odd, this will never work, so the number of ways is 0, which agrees with this formula. Now assume that k is even. If $n = 1$, then there is 1 possibility (all 1s), so

$$T_{k,n} = 1$$

for all even k. Now, assume that the formula works for some fixed value n. To count the number of strongly even sequences that use the numbers in $[n+1]$, we break it down into the cases where element $n+1$ appears i times, and count the number of possibilities for the other n numbers in the remaining $k-i$ positions in the sequence:

(i) "$n+1$" appears 0 times, so the remaining numbers appear k times. There are $T_{k,n}$ ways of doing this.

(ii) "$n+1$" appears 1 time. This is an odd number, and so there should be $T_{k-1,n} = 0$ ways of doing this.

(iii) "$n+1$" appears 2 times, so the remaining numbers appear $k-2$ times. There are $\binom{k}{2}$ ways for placing the numbers "$n+1$" and $T_{k-2,n}$ ways of choosing the remaining numbers, so there are $\binom{k}{2}T_{k-2,n}$ ways.

(iv) In general, if "$n+1$" appears j times, there are $\binom{k}{j}T_{k-j,n}$ ways. Take note that this works even for odd j as $k-j$ is odd, so that while we are counting

$$\sum_{j=0}^{k/2}\binom{k}{2j}T_{k-2j,n}$$

it is much more convenient to use the equivalent sum

$$\sum_{j=0}^{k}\binom{k}{j}T_{k-j,n}$$

despite the fact that half of its summands are zero.

So now to show that the number of strongly even sequences using the set $[n+1]$ is equal to $T_{n+1,k}$ we use some algebra:

$$\begin{aligned}
&\sum_{j=0}^{k}\binom{k}{j}T_{k-j,n}\\
=\;&\sum_{j=0}^{k}\binom{k}{j}\left[\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^{k-j}\right]\\
=\;&\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left[\sum_{j=0}^{k}\binom{k}{j}(n-2i)^{k-j}\right]\\
=\;&\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n+1-2i)^{k}\\
=\;&\frac{1}{2^{n+1}}\cdot 2\left[\binom{n}{0}(n+1)^k+\binom{n}{1}(n-1)^k+\binom{n}{2}(n-3)^k+\cdots+\binom{n}{n}(n-1)^k\right]\\
=\;&\frac{1}{2^{n+1}}\left[\binom{n}{0}(n+1)^k+\binom{n}{0}(n+1)^k+\left[\binom{n}{1}(n-1)^k+\cdots+\binom{n}{n}(n-1)^k\right]+\left[\binom{n}{1}(n-1)^k+\cdots+\binom{n}{n}(n-1)^k\right]\right]\\
=\;&\frac{1}{2^{n+1}}\left[\binom{n}{0}(n+1)^k+\binom{n}{n}(n+1)^k+\binom{n}{n-1}(n-1)^k+\cdots+\binom{n}{0}(n-1)^k+\binom{n}{1}(n-1)^k+\cdots+\binom{n}{n}(n-1)^k\right]\\
=\;&\frac{1}{2^{n+1}}\left[\binom{n}{0}(n+1)^k+\binom{n}{0}(n-1)^k+\binom{n}{1}(n-1)^k+\cdots+\binom{n}{n-1}(n-1)^k+\binom{n}{n}(n-1)^k+\binom{n}{n}(n+1)^k\right]\\
=\;&\frac{1}{2^{n+1}}\left[\binom{n+1}{0}(n+1)^k+\binom{n+1}{1}(n-1)^k+\cdots+\binom{n+1}{n}(n-1)^k+\binom{n+1}{n+1}(n+1)^k\right]\\
=\;&\frac{1}{2^{n+1}}\sum_{i=0}^{n}\binom{n+1}{i}(n+1-2i)^{k}\\
=\;&T_{k,n+1}.
\end{aligned}$$