3 | Sequences of subsets

(a) Let $n, k \geq 1$. Consider the bijection between the sequences $S_0, S_1, \ldots S_k$ of subsets of $[n]$ such that for any $1 \leq i \leq l$ we have either:

$$\begin{array}{lll} S_i \supset S_{i-1} & \text{and} & |S_i \setminus S_{i-1}| = 1, \text{ or} \\ S_i \subset S_{i-1} & \text{and} & |S_{i-1} \setminus S_i| = 1 \end{array}$$

and the set of pairs $(S_0, \{b_i\}_{i=1}^k)$.

$$S_0, S_1, \ldots S_k \longrightarrow (S_0, \{S_i \triangle S_{i-1}\}_{i=1}^k)$$

where $S_i \triangle S_{i-1}$ is the symmetric difference of the sets S_i and S_{i-1}, and whose inverse is:

$$(S_0, \{b_i\}_{i=1}^k) \longrightarrow S_0, S_1 = S_0 \triangle \{b_1\}, S_2 = S_1 \triangle \{b_2\}, \ldots, S_k = S_{k-1} \triangle \{b_k\}$$

Now, the cardinality of the set of such pairs is $2^n \times n^k$, since there are 2^n subsets of $[n]$ and n^k k-sequences of elements in $[n]$.

Then, the number of such sequences $S_0, S_1, \ldots S_k$ is:

$$2^n \times n^k$$

(b) First note that the number of such sequences with the additional condition $S_0 = S_k = \emptyset$ is zero if k is odd, but this agrees with the formula:

$$\begin{aligned} 2\sum_{i=0}^{n} \binom{n}{i} (n-2i)^k &= \sum_{i=0}^{n} \binom{n}{i} (n-2i)^k + \sum_{i=0}^{n} \binom{n}{(n-i)} (n-2(n-i))^k \\ &= \sum_{i=0}^{n} \left(\binom{n}{i} (n-2i)^k + \binom{n}{(n-i)} (n-2(n-i))^k \right) \\ &= \sum_{i=0}^{n} \binom{n}{i} \left((n-2i)^k + (2i-n)^k \right) \\ &= \sum_{i=0}^{n} \binom{n}{i} \left((n-2i)^k - (n-2i)^k \right) \\ &= \sum_{i=0}^{n} \binom{n}{i} 0 = 0 \end{aligned}$$

so $\sum_{i=0}^{n}\binom{n}{i}(n-2i)^k = 0$ and $\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^k = 0$.

(Idea: José Acevedo) Considering the bijection in part (a), the number of sequences such that $S_0 = S_k = \emptyset$ is just the number of k-sequences of elements in $[n]$ such that every element appears in an even number of positions.

If $k = a_1 + a_2 + \cdots + a_n$ is an n-weak composition of k, then there are $\binom{k}{a_1\ a_2\ \ldots\ a_n}$ sequences of length k having exactly a_1 1s, a_2 2s, and so on. So what we want to count is:

$$\sum_{\substack{a_1+a_2+\cdots+a_n=k\\ a_i\in 2\mathbb{Z}_{\geq 0}}} \binom{k}{a_1\ a_2\ \ldots\ a_n}$$

Remember that:

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{\substack{a_1+a_2+\cdots+a_n=k\\ a_i\in \mathbb{Z}_{\geq 0}}} \binom{k}{a_1\ a_2\ \ldots\ a_n} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$$

Now, we will evaluate this multinomial in all the points of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ (2^n total) and sum up.

On the left hand, there are $\binom{n}{i}$ ways to give values to the n variables in such a way that exactly i terms are -1 and the remaining $n-i$ are 1, and in this case, the evaluation is $((-1)i + (1)(n-i))^k = (n-2i)^k$. Adding over all the possible values of i, the left side is:

$$\sum_{i=0}^{n}\binom{n}{i}(n-2i)^k$$

On the right hand, first fix an element $\binom{k}{a_1\ a_2\ \ldots\ a_n} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ and evaluate it on all the elements $(\pm 1, \pm 1, \ldots, \pm 1)$ and sum. Clearly, if all a_i's are even, $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ will always be 1, so the sum becomes: $2^n \binom{k}{a_1\ a_2\ \ldots\ a_n}$. In the case that some a_j is odd, then all the evaluations in $(\pm 1, \ldots, \pm 1, 1, \pm 1, \ldots, \pm 1)$ (where the 1 appears in the j-th position) have the opposite sing that the evaluations in $(\pm 1, \ldots, \pm 1, -1, \pm 1, \ldots, \pm 1)$, so the sum is zero. Then, adding all the evaluations over the right hand gives:

$$\sum_{\substack{a_1+a_2+\cdots+a_n=k\\ a_i\in 2\mathbb{Z}_{\geq 0}}} 2^n \binom{k}{a_1\ a_2\ \ldots\ a_n} = 2^n \sum_{\substack{a_1+a_2+\cdots+a_n=k\\ a_i\in 2\mathbb{Z}_{\geq 0}}} \binom{k}{a_1\ a_2\ \ldots\ a_n}$$

So we have the equality:

$$\sum_{i=0}^{n}\binom{n}{i}(n-2i)^k = 2^n \sum_{\substack{a_1+a_2+\cdots+a_n=k\\ a_i\in 2\mathbb{Z}_{\geq 0}}} \binom{k}{a_1\ a_2\ \ldots\ a_n}$$

and therefore:

$$\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^k = \sum_{\substack{a_1+a_2+\cdots+a_n=k\\ a_i\in 2\mathbb{Z}_{\geq 0}}}\binom{k}{a_1\ a_2\ \ldots\ a_n}$$

Thus, we have proven that the number of such sequences is:

$$\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^k$$