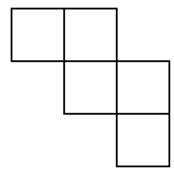
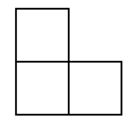
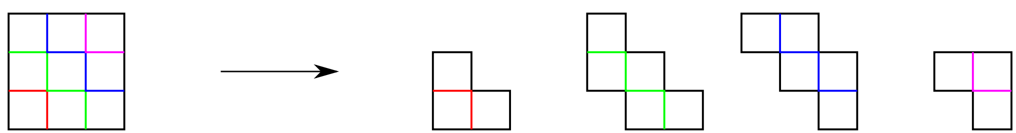
6. First, notice that an equivalent definition of an NE-path tiling of an $n \times n$ grid (note: in the following, we will assume $n > 1$) is a tiling in which each square has at most one of its North and East borders open and at most one of its South and West borders open.

We define a *South-East (SE) staircase sequence* of squares in the grid to be a sequence of squares produced by alternating South and East steps. For example,



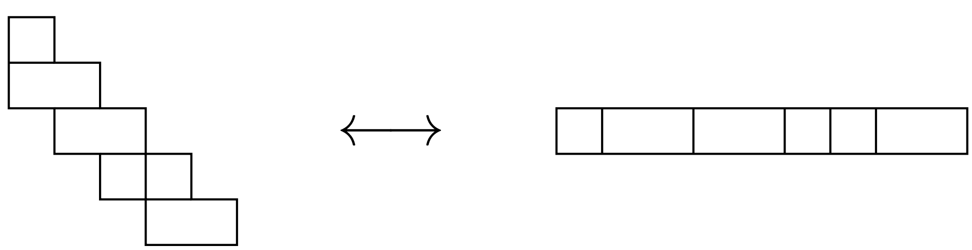


are SE staircase sequences. We may decompose an $n \times n$ grid into SE staircase sequences such that, while a given square may appear in more than one of these sequences, each interior border (i.e., each border that does not lie along the grid boundary) appears as an interior border in precisely one sequence. Below is an illustration of this decomposition for a 3×3 grid:



We observe that if a square's North and East borders are both interior borders, then they are interior borders of the same SE staircase sequence in the decomposition. Moreover, if we define interior borders of an SE staircase sequence to be consecutive if we reach one immediately after the other when traversing the sequence starting from, say, the NW-most square, then a square's North and East borders are always consecutive in the SE staircase sequence in which they reside. The same results hold for a square's South and West borders. We call a specification of the interior borders of a subgrid of an $n \times n$ grid as open or closed a border assignment for this subgrid, and we call a border assignment for an SE staircase sequence "NE-valid" if no consecutive interior borders are open. It then follows from the above observations that if the border assignments for each of the SE staircase sequences in the SE staircase decomposition of an $n \times n$ grid is NE-valid, then there is no square in the grid whose North and East borders or South and West borders are both open, so the resulting tiling must be an NE-path tiling. Further, we can generate every NE-path tiling by this procedure, since we can generate every specification of interior borders as open or closed in which no cell has both its North and East or South and West borders open. Hence, the number of NE-path tilings of an $n \times n$ grid is equal to the number of ways to make border assignments for each of its SE staircase decomposition sequences such that each assignment is NE-valid.

Observe that border assignments for distinct SE staircase sequences are independent. There are $2(n-1)$ SE staircase sequences in the decomposition for an $n \times n$ grid, and if we order them in the NE direction starting from the SW corner, then for $1 \leq k \leq n-1$, the kth sequence in the decomposition has $2k+1$ squares and is equivalent (in the sense that it is a reflection across the line $y = -x$) to the $(2n-k-1)$th sequence (as in the 3×3 example above), and thus the number of NE-valid border assignments for the kth sequence is equal to the number of NE-valid border assignments for the $(2n-k-1)$th sequence. Hence we restrict our attention to $1 \leq k \leq n-1$. We have a bijection between NE-valid border assignments for the kth SE staircase sequence and tilings of a $1 \times (2k+1)$ rectangular board by 1×1 rectangles (squares) and 1×2 rectangles (dominoes): We identify the jth square (in the SE direction) of the staircase with the jth square of the board, and declare that the jth square of the board is the right edge of a tile if and only if the interior border between the jth and $(j+1)$th square is closed or if $j = 2k+1$. Visually, the bijection simply corresponds to pulling on the ends of the staircase to straighten it out:



Tilings of a $1 \times (2k+1)$ board by squares and dominoes are in turn in bijection with tilings of a $2 \times (2k+1)$ board by dominoes (just take the upper or lower half of the board), and, as discussed in class, the number of tilings of the latter type is f_{2k+2}, the $(2k+2)$th Fibonacci number. Thus, the number of NE-valid border assignments to the kth staircase (and hence the $(2n-k-1)$th staircase) is f_{2k+2}, and it follows that if we let t_n denote the number of NE-path tilings of an $n \times n$ grid, then

$$t_n = \prod_{k=1}^{n-1} f_{2k+2}^2 = \prod_{k=2}^{n} f_{2k}^2$$