5. (Compositions with bounded parts) Let k be a fixed positive integer. For each n let $c_k(n)$ be the number of compositions of n such that every part is less than k. Prove that

$$\sum_{n\geq 0} c_k(n)x^n = \frac{1-x}{1-2x+x^k}$$

We have the recurrence relation $c_k(n) = c_k(n-1)+c_k(n-2)+\cdots+c_k(n-(k-1))$ for $n>0$ (if we suppose that if $n<0$ then $c_k(n)=0$) depending of which number is the first in the composition.

So $C(x) = \sum_{n\geq 0} c_k(n)x^n = c_k(0) + c_k(1)x + c_k(2)x^2 + \cdots + c_k(k-1)x^{k-1} + c_k(k)x^k + \ldots$

$= c_k(0) + c_k(0)x + (c_k(1)+c_k(0))x^2 + \cdots + (c_k(k-2)+c_k(k-3)+\cdots+c_k(0))x^{k-1} +$

$(c_k(k-1)+c_k(k-2)+\cdots+c_k(1))x^k + \ldots$

$\Rightarrow C(x) = 1 + C(x)(x+x^2+\cdots+x^{k-1})$

$\Rightarrow C(x)(1-x-x^2-\cdots-x^{k-1}) = 1$

$\Rightarrow C(x) = \dfrac{1}{1-x-x^2-\cdots-x^{k-1}} = \dfrac{1-x}{(1-x)(1-x-x^2-\cdots-x^{k-1})} = \dfrac{1-x}{1-2x+x^k}$