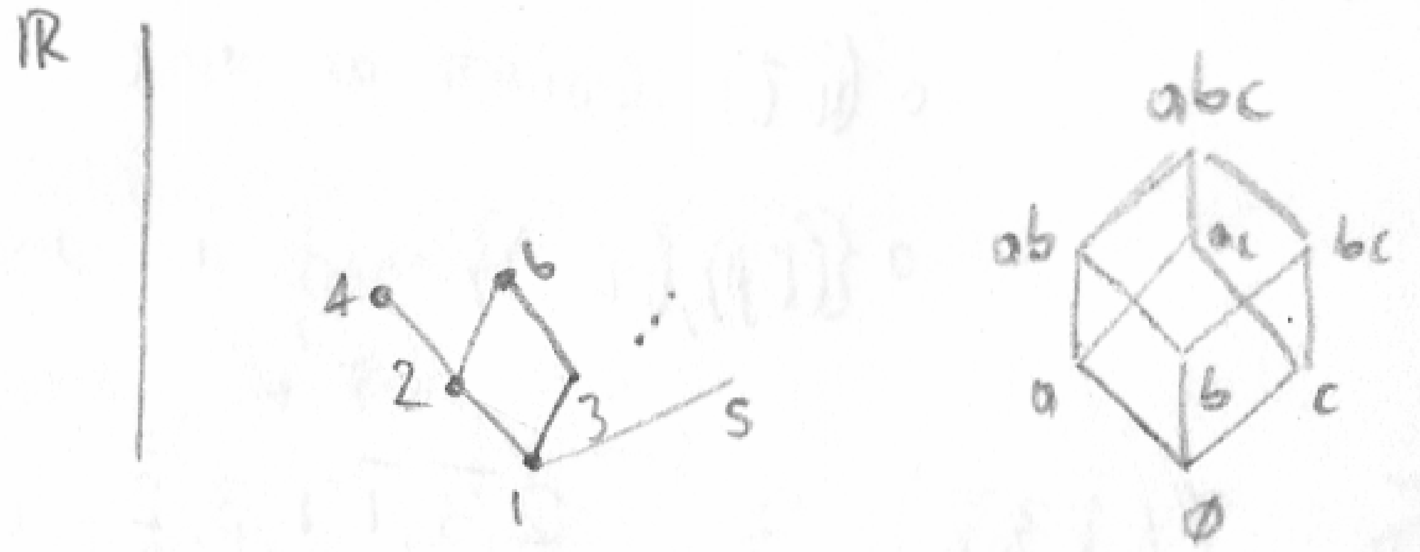
2. Bruhat Order

A **poset** is a set S with a partial order $<$ such that

- For $x \neq y$, either $x<y$ or $x>y$ or neither
- $x<y$, $y<z$ imply $x<z$.

Examples:

(We draw only the "cover rels".)



$(\mathbb{R}, <)$ $(\mathbb{Z}^{>0}, |)$ $(2^S, \subseteq)$

Def

Given (W,S), let $T = \{wsw^{-1} \mid w \in W, s \in S\}$.

Define

$u \to v$ if $v = ut$ $\ell(v) > \ell(u)$ for $t \in T$

$u \leq v$ if

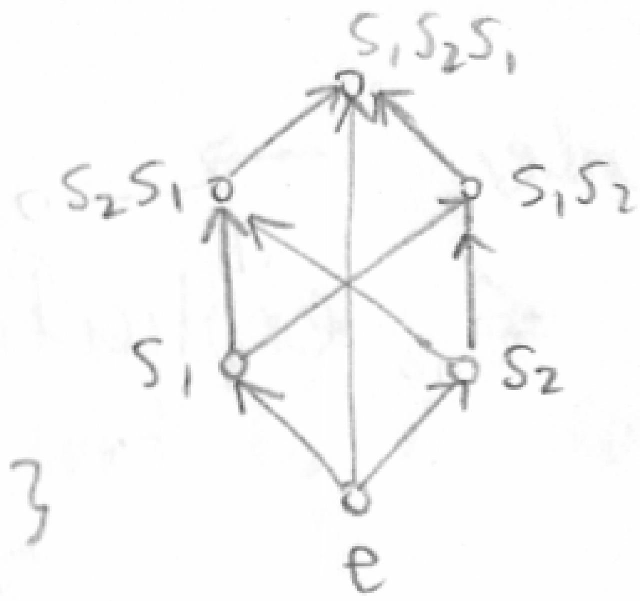
$u = w_0 \to w_1 \to \dots \to w_k = v$ for some $w_i \in W$.

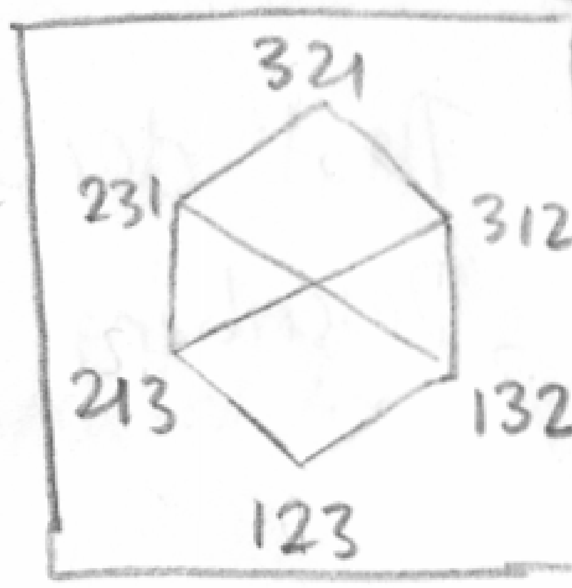
This $\leq$ is the Bruhat order of (W,S).

Ex. $W = S_3$

$S = \{s_1, s_2\}$

$T = \{s_1, s_2, s_1 s_2 s_1\}$





Note

Defined $u \to ut$ (some t)

Could have defined $u \to t'u$ (some t')

We get the same answer, since $ut = (utu^{-1})u$

The Bruhat order of S_n

Recall:

If $w \in S_n$, $\ell(w)$ = number of inversions

= # of pairs "out of order"

$\ell(14253) = 3$

Right-mult. by (ij) switches positions i and j in w

Left-mult. by (ij) switches numbers i and j in w

$14253 \cdot (24) = 15243$

$(24) \cdot 14253 = 12453$

So

$\_\_ i \_\_ j \_\_ \to \_\_ j \_\_ i \_\_$ for $i<j$.

That describes →

Describing $\leq$ explicitly is tricky.

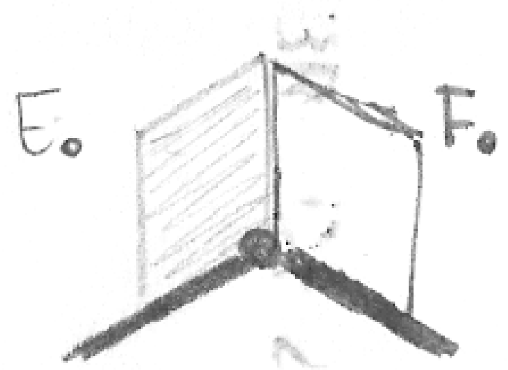
Origins: The Schubert calculus (Hilbert's 15th problem)

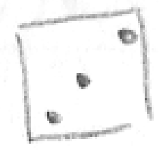
A <u>flag</u> in $\mathbb{R}^n$ is

$F_\bullet : \quad 0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{R}^n \qquad \dim F_i = i$

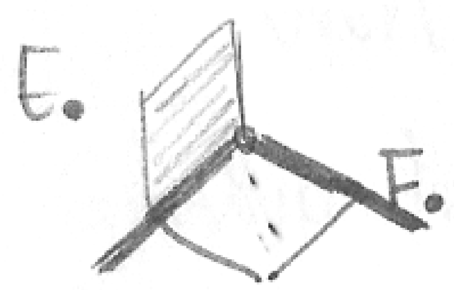
How do two flags intersect?

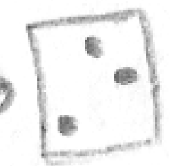
<u>rank table</u> → <u>permut</u>



$$\dim E_i \cap F_j \quad \begin{array}{c|ccc} & F_1 & F_2 & F_3 \\ \hline E_1 & 0 & 0 & 1 \\ E_2 & 0 & 1 & 2 \\ E_3 & 1 & 2 & 3 \end{array} \rightarrow$$  → 321

↓ more special



$$\begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{matrix} \rightarrow$$  → 312

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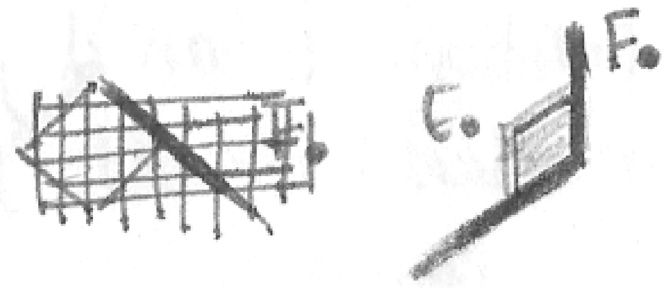
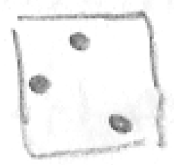


$$\begin{matrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{matrix} \rightarrow$$  → 213

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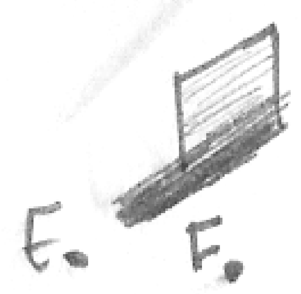
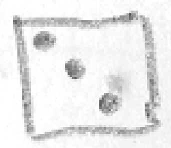


$$\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{matrix} \rightarrow$$  → 123

A chain in the Bruhat order

Fix $E_\bullet$, vary $F_\bullet$

Relative position of $F_\bullet$ wrt $E_\bullet$ is described by a permutation $w \in S_n$

"Flag variety" = space of flags (alg. var.)

"Schubert cell" C_w = flags $F_\bullet$ in position w

"Schubert variety" $\overline{C_w}$ = flags in position w or more special than w

$\boxed{\underline{\text{Thm}} \quad \overline{C_u} \subseteq \overline{C_v} \iff u \leq v \text{ in Bruhat}}$

position v is more general than u

<u>Applications</u> (Good project!)

- Topology of flag variety.
- Schubert calculus (Hilbert's 15th problem)

  How many lines in $\mathbb{R}^3$ intersect four given lines?

  (generic)