Classification of Regular Polytopes

P - n-dim polytope

A (complete) flag in P is a seq of faces

$$F_0 \subset F_1 \subset \cdots \subset F_n = P$$

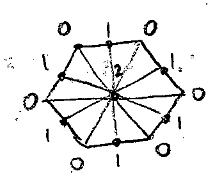
with $\dim F_i = i$.

<u>Note</u>: If you move P around, to determine the position of P it is enough to specify the position of a flag.

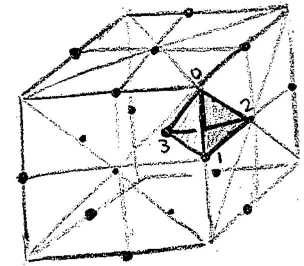
<u>Def</u> P is <u>regular</u> if there is an isometry (distance preserving lin. transf) from any flag to any other flag.
"looks the same from any flag"

<u>Barycentric Subdiv of P</u>:

- Put a vertex in the center of mass of each face
- Fill in the simplex connecting the vertices of each flag. They subdivide P.

Ex: 

(If P is regular, they are congruent)



Choose a "fundamental chamber" C

Note:

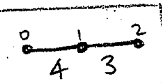
$$\begin{pmatrix}\text{chambers of}\\ \text{subdiv. of } P\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{flags}\\ \text{of } P\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{symmetries}\\ \text{of } P\end{pmatrix}$$

<u>n basic reflections</u>: $(0 \le i \le n-1)$

s_i fixes all vertices of C except i

$$\boxed{\underline{\text{Fact}}.\ \text{The group of symmetries of } P \text{ is the Coxeter gp gen by } s_0, \ldots, s_{n-1}}$$

In example, $\measuredangle 0,1 = \frac{\pi}{4}$, $\measuredangle 1,2 = \frac{\pi}{3}$, $\measuredangle 0,2 = \frac{\pi}{2}$

cube $\longrightarrow$ W = 

$$\boxed{\underline{\text{Fact}}\ \measuredangle i,j = \pi/2 \text{ for } |i-j|>1}$$

<u>Ex</u>, why $\measuredangle 0,2 = \pi/2$? Let F_1 be the face of 1.

- s_2 fixes $F_1 \rightarrow \alpha_2 \perp F_1$
- s_0 sends 0 to 0' on F_1 $\rightarrow 00' \in F_1$
  $\rightarrow \alpha_1 \in F_1$

  $\Rightarrow \alpha_1 \perp \alpha_2$

So:

- The group of symmetries of a regular polytope is a finite Coxeter group
- The Coxeter diagram is a path

So it is one of:

A_n	$0 - 1 - \cdots - n$	simplex
B_n	$0 \overset{4}{-} 1 - \cdots - n$	hypercube
	$0 - 1 - \cdots \overset{4}{-} n$	crosspolytope
H_3	$0 \overset{5}{-} 1 - 2$	icosahedron
	$0 - 1 \overset{5}{-} 2$	dodecahedron
F_4	$0 - 1 \overset{4}{-} 2 - 3$	24-cell
H_4	$0 \overset{5}{-} 1 - 2 - 3$	120-cell
	$0 - 1 - 2 \overset{5}{-} 3$	600-cell

The Coxeter diagram determines the angles between the facets of the fundamental chamber which in turn determines the polytope.

So these are all the regular polytopes!