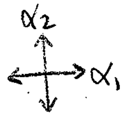
These can be obtained in rk 2:

$a_{12}a_{21}$	0	1	2	3	≥ 4
m_{12}	2	3	4	6	∞

(Note: $\dfrac{a_{12}}{a_{21}} = \dfrac{\langle\alpha_1,\alpha_1\rangle}{\langle\alpha_2,\alpha_2\rangle}$)

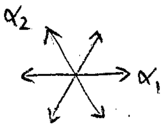
- $a_{12}a_{21}=0$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



- $a_{12}a_{21}=1$

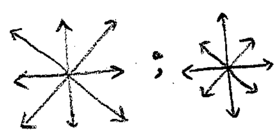
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$



Sup $\langle\alpha_1,\alpha_1\rangle = 1 = \langle\alpha_2,\alpha_2\rangle$
$\langle\alpha_1,\alpha_2^\vee\rangle = -1$
$\langle\alpha_1,\alpha_2\rangle = -1/2$
$\cos\theta = -1/2$

- $a_{12}a_{21}=2$

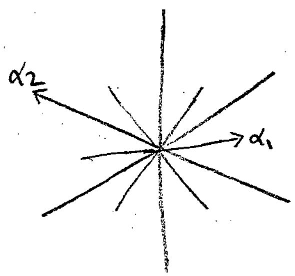
$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$



Sup $\langle\alpha_1,\alpha_1\rangle = 1$
$\langle\alpha_2,\alpha_2\rangle = 2$
$\langle\alpha_1,\alpha_2^\vee\rangle = -1$
$\langle\alpha_1,\alpha_2\rangle = -1$
$\cos\theta = -1/\sqrt{2}$

- $a_{12}a_{21}=3$

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$



$\langle\alpha_1,\alpha_1\rangle = 1$
$\langle\alpha_2,\alpha_2\rangle = 3$
$\langle\alpha_1,\alpha_2^\vee\rangle = -1$
$\langle\alpha_1,\alpha_2\rangle = -3/2$
$\cos\theta = -\sqrt{3}/2$

- $a_{12}a_{21}\geq 4$:

$$\begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix} \to \Delta = \{\alpha_1,\alpha_2\}$$
$\Phi = W\Delta$
$(s_1, s_2 = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ a & -1 \end{bmatrix})$
Check root system axioms! (Exercise)

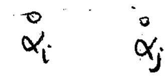
need:
- $\langle\alpha_1,\alpha_1\rangle = a$
- $\langle\alpha_2,\alpha_2\rangle = b$
- $\langle\alpha_1,\alpha_2^\vee\rangle = -a$
- $\langle\alpha_1,\alpha_2\rangle = -\dfrac{ab}{2}$

(Impossible in Euclidean geom by Cauchy: $\langle\alpha_1,\alpha_1\rangle\langle\alpha_2,\alpha_2\rangle \geq \langle\alpha_1,\alpha_2\rangle^2$)

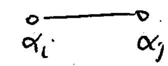
Dynkin diagram:

vertices: simple roots $\alpha_1,\dots,\alpha_n$

edges:

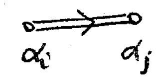


if $a_{ij} = a_{ji} = 0$

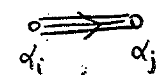


if $a_{ij} = a_{ji} = -1$

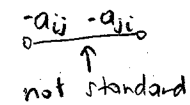
(arrows long → short)



if $a_{ij} = -2$ $a_{ji} = -1$

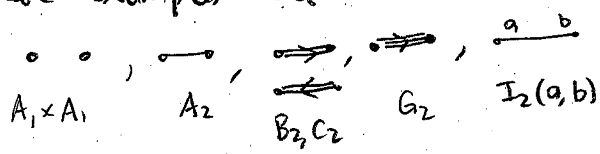


if $a_{ij} = -3$, $a_{ji} = -1$



if $a_{ij}a_{ji} \geq 4$

These example are



Theorem. An integer square matrix is the Cartan matrix of a crystallographic root system if and only if:

- $a_{ii} = 2$
- $a_{ij} \leq 0$ $i \neq j$
- $a_{ij} = 0 \iff a_{ji} = 0$
- Along every circuit $i_1 i_2 \cdots i_k$ of the Dynkin diagram,

$$\frac{a_{i_1 i_2}}{a_{i_2 i_1}} \cdot \frac{a_{i_2 i_3}}{a_{i_3 i_2}} \cdot \ldots \cdot \frac{a_{i_k i_1}}{a_{i_1 i_k}} = 1$$